Complex analysis, homework 10, solutions.

Exercise 1. [8 points] For $n \geq 0$, let

$$z_n = \frac{(n+i)^2 - 2in^2}{n^2}.$$

Prove that $\lim_{n\to\infty} z_n = 1 - 2i$ using the definition of the limit.

Solution. First note that for any $n \geq 1$,

$$\frac{(n+i)^2 - 2in^2}{n^2} = \frac{n^2 + 2in - 1 - 2in^2}{n^2} = 1 - \frac{1}{n^2} - 2i + \frac{2i}{n}.$$

Let $\varepsilon > 0$. Let n_0 be the first integer larger than $\sqrt{5}/\varepsilon$. For any $n \geq n_0$, we have

$$|z_n - (1-2i)| = \left| -\frac{1}{n^2} + \frac{2i}{n} \right| = \sqrt{\frac{1}{n^4} + \frac{4}{n^2}} \leq \sqrt{\frac{1}{n^2} + \frac{4}{n^2}} = \frac{\sqrt{5}}{n},$$

where we used that $n^{-4} \leq n^{-2}$ for $n \geq 1$. Since $n \geq n_0 > \sqrt{5}/\varepsilon$, we get

$$|z_n - (1-2i)| < \varepsilon.$$

This proves $\lim_{n\to\infty} z_n = 1 - 2i$.

Exercise 2. [6 points] Let $(z_n)_{n\geq 0}$ be a sequence of complex numbers. Let $S \in \mathbb{C}$. Prove that

$$\sum_{n=0}^{\infty} z_n = S \quad \Rightarrow \quad \sum_{n=0}^{\infty} \overline{z_n} = \overline{S}.$$

Solution. We write $z_n = x_n + iy_n$ and $S = X + iY$. Since $\sum_{n=0}^{\infty} z_n = S$, by the theorem of Section 61, we know that

$$\sum_{n=0}^{\infty} x_n = X \quad \text{and} \quad \sum_{n=0}^{\infty} y_n = Y.$$

Therefore, we have

$$\sum_{n=0}^{\infty} x_n = X \quad \text{and} \quad \sum_{n=0}^{\infty} (-y_n) = -Y.$$

Note that $\overline{z_n} = x_n + i(-y_n)$. Hence, applying again the theorem of Section 61 (but in the other direction), we get that

$$\sum_{n=0}^{\infty} \overline{z_n} = X + i(-Y) = \overline{S}.$$

Exercise 3. [10 points] Prove that the Taylor series of Log at i is

$$\text{Log}(z) = \frac{i\pi}{2} + \sum_{k=1}^{\infty} \frac{-i^k}{k}(z-i)^k.$$

Precise the complex numbers z for which this formula applies.

Solution. The function Log is analytic on $\mathbb{C}\setminus\mathbb{R}_-$, so the largest open disk centered at i on which Log is analytic has radius 1. So Taylor's theorem tells us that, for any $|z-i|<1$,

$$\mathrm{Log}(z)=\sum_{k=0}^{\infty}\frac{\mathrm{Log}^{(k)}(i)}{k!}(z-i)^k.$$

We prove by induction on $n\geq 1$ that, for any $z\in\mathbb{C}\setminus\mathbb{R}_-$, $\mathrm{Log}^{(n)}(z)=(-1)^{n+1}(n-1)!z^{-n}$

- **Basis step:** For any $z\in\mathbb{C}\setminus\mathbb{R}_-$, $\mathrm{Log}'(z)=z^{-1}=(-1)^{n+1}(n-1)!z^{-n}$ with $n=1$.
- **Inductive step:** Let $k\geq 1$, assume the formula is known for $n=k$. Then we have

$$\mathrm{Log}^{(k+1)}(z)=\frac{\mathrm{d}}{\mathrm{d}z}(-1)^{k+1}(k-1)!z^{-k}=(-1)^{k+1}(k-1)!\cdot(-k)z^{-k-1}=(-1)^{k+2}k!z^{-(k+1)},$$

which proves the formula for $n=k+1$.

Hence the formula is proved by induction and, noting that $\mathrm{Log}(i)=i\pi/2$, we get, for any $|z-i|<1$,

$$\mathrm{Log}(z)=\frac{i\pi}{2}+\sum_{k=1}^{\infty}\frac{(-1)^{k+1}(k-1)!i^{-k}}{k!}(z-i)^k=\frac{i\pi}{2}+\sum_{k=1}^{\infty}\frac{-i^{2k}i^{-k}}{k}(z-i)^k,$$

so finally

$$\mathrm{Log}(z)=\frac{i\pi}{2}+\sum_{k=1}^{\infty}\frac{-i^{k}}{k}(z-i)^k.$$

Exercise 4. [6 points] Find the Taylor series at 0 of

$$f(z)=\frac{\sin(z)-z}{z^2}.$$

Solution. Using the Taylor series of $\sin z$, for any $z\neq 0$, we have

$$\begin{aligned}
f(z)&=\frac{1}{z^2}\left(\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}z^{2n+1}-z\right)\\
&=\frac{1}{z^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n+1)!}z^{2n+1}\\
&=\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n+1)!}z^{2n-1}\\
&=\sum_{k=0}^{\infty}\frac{(-1)^{k+1}}{(2k+3)!}z^{2k+1},
\end{aligned}$$

setting $k=n-1$.