Complex analysis, homework 12 due May 4th.

Exercise 1.[8 points] Find the singularities of the following functions and the residues of the function at each singularity.

(1) $f(z) = \dfrac{1}{z^2+5z+6}$; (2) $f(z) = \dfrac{1}{(z^2-1)^2}$.

Exercise 2.[10 points] Evaluate the integral $\int_C f(z)\,\mathrm{d}z$ for the following functions and where C is a positively oriented simple closed contour around 0.

(1) $f(z) = z^7 \cos\left(\dfrac{1}{z^2}\right)$; (2) $f(z) = \dfrac{\sinh(2z) - 2z}{z^8}$.

Exercise 3.[4 points] Let f be an entire function such that for any $\theta \in [0, \pi]$, $f(i\theta) = e^{\theta}$. Find $f(z)$ for any $z \in \mathbb{C}$. Justify your answer.

Exercise 4.[8 points] Let f be the function defined by

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} b_n (z-z_0)^{-n},$$

where we assume that these series converge on the annular domain $D = \{R_1 < |z-z_0| < R_2\}$ for some $0 \leq R_1 < R_2 \leq +\infty$. The goal of this exercise is to prove the following result seen in class: f is analytic on D and

$$f'(z) = \sum_{n=1}^{\infty} n a_n (z-z_0)^{n-1} + \sum_{n=1}^{\infty} (-n) b_n (z-z_0)^{-n-1}, \qquad z \in D.$$

For this, you are allowed to apply the theorem for power series seen in Section 71 (involving only non-negative powers of $z - z_0$), but not the theorem for Laurent series that we are trying to prove.

(1) Let $f_1(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$, for $z \in D$. Explain why this power series has a radius of convergence larger than or equal to R_2. Deduce that f_1 is analytic on D and

$$f_1'(z) = \sum_{n=1}^{\infty} n a_n (z-z_0)^{n-1}, \qquad z \in D.$$

(2) Let $f_2(z) = \sum_{n=1}^{\infty} b_n (z-z_0)^{-n}$, for $z \in D$. We introduce the following power series

$$g(w) = \sum_{n=1}^{\infty} b_n w^n.$$

Noting that for $z \in D$, $f_2(z) = g(\frac{1}{z-z_0})$, show that the power series $g(z)$ has a radius of convergence larger than or equal to $1/R_1$ and therefore is analytic on $\{w : |w| < 1/R_1\}$. Find $g'(w)$.

(3) Deduce from question (b) that f_2 is analytic on D and

$$f_2'(z) = \sum_{n=1}^{\infty} (-n) b_n (z - z_0)^{-n-1}, \qquad z \in D.$$

(4) Using questions (a) and (c), conclude the exercise.