Complex analysis, homework 10 due April 20th.

Exercise 1. [8 points] For $n \geq 0$, let

$$z_n = \frac{(n+i)^2 - 2in^2}{n^2}.$$

Prove that $\lim_{n\to\infty} z_n = 1 - 2i$ using the definition of the limit.

Exercise 2. [6 points] Let $(z_n)_{n\geq 0}$ be a sequence of complex numbers. Let $S \in \mathbb{C}$. Prove that

$$\sum_{n=0}^{\infty} z_n = S \quad \Rightarrow \quad \sum_{n=0}^{\infty} \overline{z_n} = \overline{S}.$$

Exercise 3. [10 points] Prove that the Taylor series of Log at i is

$$\mathrm{Log}(z) = \frac{i\pi}{2} + \sum_{k=1}^{\infty} \frac{-i^k}{k}(z-i)^k.$$

Precise the complex numbers z for which this formula applies.

Exercise 4. [6 points] Find the Taylor series at 0 of

$$f(z) = \frac{\sin(z) - z}{z^2}.$$