Analytic number theory, homework 4.

Exercise 1. Prove that, for each integer $n > 0$, the number of primes smaller than x such that $p + 2n$ is prime is $\mathrm{O}_n(x/(\log x)^2)$. Prove that

$$\sum_{p\in\mathcal{P}:p+2\in\mathcal{P}} \left(\frac{1}{p} + \frac{1}{p+2}\right) < \infty.$$

Exercise 2. Prove that, for each $\sigma > 1$,

$$\liminf_{t\to\infty} |\zeta(\sigma + it)| = \frac{\zeta(2\sigma)}{\zeta(\sigma)}.$$
$$\limsup_{t\to\infty} |\zeta(\sigma + it)| = \zeta(\sigma).$$

Exercise 3. Let $\mathcal{A}$ be a set of distinct real numbers and define, for any $\mu \in \mathcal{A}$, $\delta(\mu) = \inf_{\nu\neq\mu,\nu\in\mathcal{A}} |\nu - \mu|$. To complete the proof of the Montgomery-Vaughan inequality, prove that for any $k \in \mathbb{N}^*$,

$$\sum_{\nu\in\mathcal{A},\nu\neq\mu} \frac{\delta(\nu)}{(\mu-\nu)^k} \ll_k \delta(\mu)^{1-k}.$$
$$\sum_{\nu\in\mathcal{A},\nu\neq\mu_1,\nu\neq\mu_2} \frac{\delta(\nu)}{(\mu_1-\nu)^2(\mu_2-\nu)^2} \ll \frac{\delta(\mu_1)^{-1} + \delta(\mu_2)^{-1}}{(\mu_1-\mu_2)^2},\ \mu_1 \neq \mu_2.$$

Exercise 4. Let χ be a character mod q. Find the asymptotics for

$$\int_0^t |L_\chi(\frac{1}{2} + \mathrm{i}s)|^2 \mathrm{d}s.$$

How does the error term depend on q?

Exercise 5. Prove that for any $\varepsilon > 0$ there exists $x_0 = x_0(\varepsilon)$ such that for any m and N coprime, for any $x > \max(N, x_0)$,

$$|\{p \in \mathcal{P} \mid p \leq x,\ p = m (\mathrm{mod}\ N)\}| \leq \frac{(2+\varepsilon)x}{\varphi(N)\log(2x/N)}.$$