Analytic number theory, homework 2.

Exercise 1. Let $p_1 < p_2 < \dots$ denote the prime numbers. Show that

$$p_n = n\left(\log n + \log\log n - 1 + \frac{\log\log n}{\log n} - \frac{2}{\log n} + \mathrm{O}\left(\frac{(\log\log n)^2}{(\log n)^2}\right)\right).$$

Exercise 2. Let $Q(x)$ denote the number of square-free integers not exceeding x. Prove that

$$Q(x) = \frac{6}{\pi^2}x + \mathrm{O}(x^{1/2}).$$

Exercise 3. Let $\lambda(n) = (-1)^{\Omega(n)}$. Show that, for any $\sigma > 1$,

$$\sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s} = \frac{\zeta(2s)}{\zeta(s)}.$$

Using the method of the analytic proof of the prime number theorem, show that

$$\sum_{n\leq x} \lambda(n) \ll xe^{-c\sqrt{\log x}}.$$

Exercise 4. From the estimate

$$\sum_{n\leq x} \frac{\Lambda(n)}{n} = \log x + \mathrm{O}(1),$$

explain why $\lim_{x\to\infty} \Psi(x)/x = 1$, provided that this limit exists.

Exercise 5. Show that

$$2^{\omega(n)} = \sum_{d^2 m = n} \mu(d)\mathrm{d}(m).$$

You can consider prime powers first and then conclude by multiplicativity. Prove the asymptotics

$$\sum_{n\leq x} 2^{\omega(n)} = \frac{6}{\pi^2}x\log x + cx + \mathrm{O}(\sqrt{x}\log x).$$

Exercise 6. Show that for any $\sigma > 1$ we have

$$\left(\frac{\zeta'}{\zeta}\right)' + \left(\frac{\zeta'}{\zeta}\right)^2 = \frac{\zeta''}{\zeta}.$$

Prove that the above identity implies $\sum_{d|n} \Lambda_2(d) = (\log n)^2$, where $\Lambda_2(n) = \Lambda(n)\log n + \sum_{bc=n} \Lambda(b)\Lambda(c)$.