Algebraic combinatorics, homework 5.

Exercise 1. Prove that the number of standard Young tableaux of shape (n, n) equals the Catalan number C_n. Prove that the number of permutations $\pi \in \mathcal{S}_n$ with longest decreasing subsequence of length at most two equals the Catalan number C_n.

Exercise 2. We abbreviate $e_k(n)$ for $e_k(x_1, \dots, x_n)$, and similarly for the complete symmetric functions h_k. Prove that

$$e_k(n) = e_k(n-1) + x_n e_{k-1}(n-1),\ \ h_k(n) = h_k(n-1) + x_n h_{k-1}(n).$$

The Stirling number of the first kind $c_{n,k}$ can be defined as the number of elements in $\mathcal{S}_n$ with k disjoint cycles. The Stirling number of the second kind $S_{n,k}$ can be defined as the number of partitions of the set $[\![1, n]\!]$ into k subsets. Prove that

$$c_{n,k} = c_{n-1,k-1} + (n-1)\, c_{n-1,k},\ \ S_{n,k} = S_{n-1,k-1} + k\, S_{n-1,k}.$$

Prove that

$$\binom{n}{k} = e_k(1^n) = h_k(1^{n-k+1}),\ c_{n,k} = e_{n-k}(1, 2, \dots, n-1),\ S_{n,k} = h_{n-k}(1, 2, \dots, k).$$

Exercise 3. Prove the following determinantal identity:

$$\det\left((x_i + a_n) \dots (x_i + a_{j+1})(x_i + b_j) \dots (x_i + b_2)\right)_{i,j=1}^n = \prod_{1 \le i < j \le n} (x_i - x_j) \prod_{2 \le i \le j \le n} (b_i - a_j).$$

Explain why it generalizes the Vandermonde identity.

Exercise 4. Prove that $s_n(x_1, \dots, x_n) = h_n(x_1, \dots, x_n)$.

Exercise 5. Prove that any character of $\mathcal{S}_n$ is an integer-valued function.

Exercise 6. Let $p_k(x_1, \dots, x_r) = \sum_{j=1}^r x_j^k$, and for $\lambda = (\lambda_1, \dots, \lambda_\ell) \in \mathrm{Par}(m)$, $p_\lambda = \prod_{i=1}^\ell p_{\lambda_i}$. Prove that, in the language of Pólya's theory,

$$e_n(x_1, \dots, x_r) = Z(\mathcal{S}_n; p_1, -p_2, \dots, (-1)^{n-1} p_n).$$

Prove that $\{p_\lambda, \lambda \in \mathrm{Par}(m)\}$ is a basis for $\Lambda^m(X)$.

Exercise 7. Let $\alpha = (\alpha_1, \dots, \alpha_n)$, $\beta = (\beta_1, \dots, \beta_n)$. Prove the Giambelli identity,

$$s_{(\alpha|\beta)} = \det(s_{(\alpha_i|\beta_j)})_{i,j=1}^n,$$

where we use the Frobenius notation for partitions.