Algebraic combinatorics, homework 4.

Exercise 1. Let c_n be the number of set partitions of $[\![n]\!]$ such that each block has even number of elements. Show that the exponential generating function for c_n is equal to

$$e^{(\cosh x)-1}.$$

Exercise 2. Let $\tau(n) = |\{d \in \mathbb{N} : d \mid n\}|$.

a) Prove that, for $\mathfrak{Re}(s) > 1$,

$$\sum_{n\geq 1} \frac{\tau(n)}{n^s} = \zeta(s)^2,$$

where ζ is the Riemann zeta function.

b) Give two different proofs that, as $x \to \infty$,

$$\sum_{n\leq x} \tau(n) \sim x \log x.$$

Exercise 3. Let P be a finite partially ordered set. If $x < y$, a sequence $x = x_0 < x_1 < \cdots < x_k = y$ is called a *chain* of length k from x to y. Let $c_k(x, y)$denote the number of such chains (so $c_1(x, y) = 1$). Prove that the Möbius function of this poset satisfies

$$\mu(x, y) = \sum_{k\geq 1} (-1)^k c_k(x, y).$$

Exercise 4. Let $c_{n,i,k}$ be the number of graphs on n vertices, with i edges and k components. Show that

$$\sum_{n,k,i\geq 0} c_{n,i,k}\alpha^i \beta^k \frac{z^n}{n!} = \left(\sum_{n\geq 0} (1+\alpha)^{\binom{n}{2}} \frac{z^n}{n!}\right)^{\beta}.$$

Exercise 5. Prove that the following two sets of partitions have the same size:

(i) the partitions of n in which the even summands appear at most once;

(ii) the partitions of n for which every summand appears at most three times.

Exercise 6. Let $f(n, q)$ be the number of permutations of $[\![1, n]\!]$ whose cycles all have length $> q$.

a) Prove that the exponential generating function of $(f(n, q))_{n\geq 0}$ is

$$\sum_{n\geq 0} \frac{f(n,q)}{n!} z^n = e^{\sum_{n>q} \frac{z^n}{n}} = \frac{e^{-\left(z+\frac{z^2}{2}+\cdots+\frac{z^q}{q}\right)}}{1-z}.$$

b) Prove that, as $n \to \infty$,

$$f(n, q) \sim e^{-H_q}\, n!$$

where $H_q = 1 + \frac{1}{2} + \cdots + \frac{1}{q}$.

Exercise 7. Remember the recurrence $S_{n,k} = S_{n-1,k-1} + kS_{n-1,k}$ for the Stirling numbers. Prove that

$$\det(S_{m+i,j})_{i,j=1}^{n} = (n!)^m,$$

where $m \geq 0$, $n \geq 1$. Hint: use the Gessel-Viennot Lemma and a graph corresponding to the recurrence.

Exercise 8. Let G be the hypercube graph on 2^n vertices, that is, the vertices are the binary strings of length n and there is an edge between two vertices if they differ in exactly one coordinate. Find the number of spanning trees of G.

Hint: Fix a vertex v in G. If w is another vertex of G, let $v \cdot w$ be the dot product of v and w when they are viewed as vectors in $\mathbb{Z}^n$. Show that the column vector whose w-th coordinate is $v \cdot w$ is a right eigenvector for the Laplacian of G.

Exercise 9. Prove that $\left\{ \frac{\varphi(n)}{n}, n \geq 1 \right\}$ is dense in $[0, 1]$.

Exercise 10. A rectangle is called *good* if it has at least one side length in $\mathbb{N}$. A puzzle consists in rectangular pieces, all of them being good. Together, they form a partition of a large rectangle. Prove that the large rectangle is good.