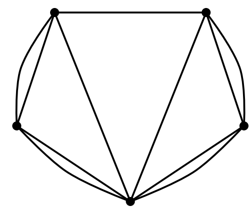
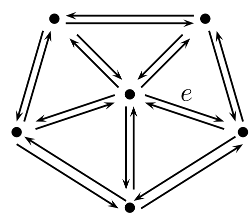
Algebraic combinatorics, homework 3.

Exercise 1. For any given $x > 0$, what is $\sum_{d \geq 1} \mu(d) \lfloor \frac{x}{d} \rfloor$?

Exercise 2. What are the adjacency matrix and the Laplacian matrix for this graph? How many spanning trees can you find? Are there Eulerian tours?



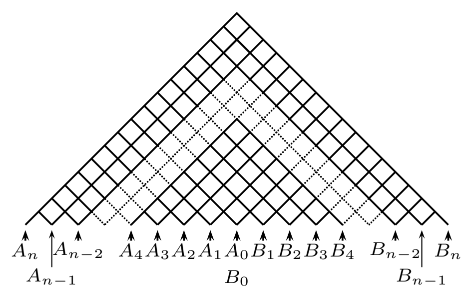
Exercise 3. What are the adjacency matrix and the Laplacian matrix for this directed graph? How many spanning trees, rooted at the central vertex, can you find? Are there Eulerian tours? How many are there starting with edge e?



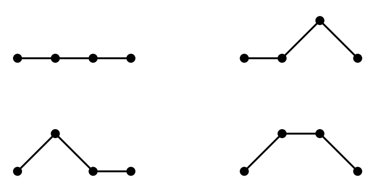
Exercise 4. Let C_{i+j} be the number of paths from A_i to B_j in the joint directed graph, where all edges go from left to right. Prove that $C_{n+1} = \sum_{i=0}^{n} C_i C_{n-i}$. Prove that

$$C_n = \frac{\binom{2n}{n}}{n+1}.$$

Denote by $H_n = (C_{i+j})_{1 \leq i,j \leq n}$ the Hankel matrix corresponding to the Calalan numbers C_k's. Prove that $\det H_n = 1$.



Exercise 5. The Motzkin number M_n is the number of lattice paths from $(0, 0)$ to $(n, 0)$ with horizontal steps and diagonal steps up or down, staying above $y = 0$. For example, $M_3 = 4$, corresponding to the joint paths. prove that $\det H_n = 1$ where H_n is the Hankel matrix corresponding to the sequence $(M_n)_{n \geq 0}$.



Exercise 6. Let r, s be two positive integers and $S = \{(a_i, b_i) \mid 1 \leq i \leq n\}$ be a set of lattice points with $0 \leq a_1 \leq \cdots \leq a_n \leq r$, $0 \leq b_1 \leq \cdots \leq b_n \leq s$. Count the number of lattice paths from $(0, 0)$ to (r, s) with steps up or right that avoid the set S.