Algebraic combinatorics, homework 2.

Exercise 1. Let P be a finite poset and $d : P \to P$ a bijection preserving the order (if $x \leq y$ then $f(x) \leq f(y)$). Show that f^{-1} preserves the order. Prove that this last statement is false if we do not assume that P is finite.

Exercise 2. For a given $k \geq 1$, let f_n be the number of rooted k-ary trees (i.e. trees in which each node has 0 or k children) with n nodes (convention: $f_0 = 1$). We note $f(x) = \sum_{n\geq 0} f_n x^n$. Prove that

$$f(x) = 1 + x f(x)^k.$$

Give a simple expression for f_n.

Exercise 3. Let $\mathcal{N}_n$ be the set of circular sequences (of 0's and 1's) of length n[1]. Let $\mathcal{M}_d$ be the number of circular sequences of length d that are not periodic[2]. Prove that

$$|\mathcal{N}_n| = \sum_{d|n} |\mathcal{M}_d|.$$

Independently, show that

$$\sum_{d|n} d|\mathcal{M}_d| = 2^n.$$

Conclude that

$$|\mathcal{N}_n| = \frac{1}{n} \sum_{d|n} \phi\left(\frac{n}{d}\right) 2^d,$$

where ϕ is Euler's totient function.

Exercise 4. Let L be a finite lattice, and $f(a, b)$ be a function from L^2 to $\mathbb{R}$. Let

$$F(a, b) = \sum_{c \leq a} f(c, b).$$

Prove that

$$\det \left(F(a \wedge b, b)\right)_{a,b\in L} = \prod_{x\in L} f(x, x).$$

Deduce that

$$\det(\gcd(i, j))_{i,j=1}^n = \prod_{k=1}^n \phi(k).$$

[1]More precisely, if

$$\tau_n : \left\{ \begin{array}{ccc} \{0,1\}^n & \to & \{0,1\}^n \\ (a_1, a_2, \dots, a_n) & \mapsto & (a_n, a_1, \dots, a_{n-1}) \end{array} \right.$$

then a and b in $\{0,1\}^n$ are considered to be the same element in $\mathcal{N}_n$ if $\tau_n^k a = b$ for some $k \geq 0$.

[2]In other words, $\mathcal{M}_d$ is the set of elements $a \in \{0,1\}^d$, identified up to the above shift, such that $\tau_d^k a = a$ implies $d \mid k$

Exercise 5. Let N_d be the number of monic irreducible polynomials of degree d over the finite field $\mathbb{F}_q$ with q elements. Prove that

$$\frac{1}{1-qx} = \prod_{d=1}^{\infty} \left(\frac{1}{1-x^d} \right)^{N_d}.$$

Conclude that $\frac{q^n}{n} = \sum_{d|n} N_d \frac{1}{n/d}$, and

$$N_d = \frac{1}{n} \sum_{d|n} \mu \left(\frac{n}{d} \right) q^d.$$