Algebraic combinatorics, homework 1.

Exercise 1. Give simpler expressions for

$$\sum_{k=0}^{n} k\binom{n}{k}^2, \quad \sum_{k=0}^{n}(-1)^k\binom{m}{k}\binom{m}{n-k},$$

by algebraic proofs (i.e. using generating functions). Thanks to a bijective argument, prove that

$$\sum_{k=0}^{n}\binom{n}{k}^2 = \binom{2n}{n}.$$

Exercise 2. Let m, n be positive integers. How many paths are there from $(0,0)$ till (m,n) with steps either $(0,1)$ or $(1,0)$?

Suppose now the step (1,1) is allowed as well. What is the number f_n of paths from $(0,0)$ till (n,n)? Prove that ($f_0 = 1$ is a convention)

$$\sum_{n\geq 0} f_n x^n = \frac{1}{\sqrt{1-6x+x^2}}.$$

Exercise 3. Let a_n be the number of subsets K of $[\![1,n]\!]$ such that, for any $1 \leq j \leq n-1$, j and $j+1$ are not both in K, and 1 and n are not both in K. By convention $a_0 = 2$ and $a_1 = 1$ (the empty subset is the only solution: as $n=1$, $a_n = a_1$). Prove that, for any $n \geq 2$,

$$a_n = a_{n-1} + a_{n-2}.$$

Give a rational expression for $\sum_{k\geq 0} a_k x^k$. What is the limit of $a_n^{1/n}$ as $n \to \infty$?

Exercise 4. For a given $k \geq 2$, prove that the number of partitions of n in which every part appears at most $k-1$ times is equal to the number of partitions for which every part is not divisible by k.

Exercise 5. Let $a_1, \dots, a_t$ be positive integers (not necessarily distinct) with greatest common divisor 1. Let b_n denote the number of solutions of

$$a_1x_1 + \cdots + a_tx_t = n$$

in non-negative integers $x_1, \dots, x_t$. What is the generating function $\sum_{n\geq 0} b_n x^n$? Prove that $b_n \sim b\, n^{t-1}$ for some coefficient $b > 0$ you will identify.

Exercise 6. Prove that for $n \geq 1$ there are 2^{n-1} compositions of n. Show in the list of all of these compositions of $n \geq 4$, the number 3 appears $n2^{n-5}$ times.

Exercise 7, bonus Find an exponential generating function and an explicit formula for the number of involutions in the symmetric group, that is, permutations π of $[\![1,n]\!]$ such that $\pi^2 = \mathrm{Id}$.