Algebra practice problems
Hints and solutions

Note: we do not give solutions to the questions where one needs to prove that something is a (normal) subgroup. The procedure is always the same, one should check the three axioms SG1, SG2, SG3 (closure, identity, inverses). To check that it is normal, use the definition.

Exercise 1. Let $G$ be a group and let $H_1, H_2$ be normal subgroups of $G$. Show that $H_1 \cap H_2$ is a normal subgroup of $G$.

Exercise 2. Let $H$ be a subgroup of a group $G$. The centralizer of $H$ in $G$ is defined to be the set $C_H(G) = \{x \in G, \; xh = hx \text{ for all } h \in H\}$.

1. Show that $C_H(G)$ is a subgroup of $G$.

2. Show that if $H$ is normal, then $C_H(G)$ is normal.

   Solution. Assume that $H$ is normal. We need to prove that for every $x \in C \cap H(G)$ and for every $g \in G$, $gxg^{-1} \in C_H(G)$. For this, by definition of $C_H(G)$, we need to prove that for every $h \in H$,
   \[ gxg^{-1}h = hgx^{-1}. \] (1)

   Now, since $h$ is normal, we have $h^{-1}hg \in H$, and therefore, since $x \in C_H(G)$, by definition of $C_H(G)$, we have that
   \[ x(h^{-1}hg) = (h^{-1}hg)x. \]

   Multiplying by $g$ on the left and by $g^{-1}$ on the right, we get equality [1].

Exercise 3. 1. Find a permutation $\sigma \in \mathfrak{S}_9$ such that $\sigma(1, 2)(3, 4)\sigma^{-1} = (5, 6)(3, 1)$.

   Solution. We have
   \[ \sigma(1, 2)(3, 4)\sigma^{-1} = \sigma(1, 2)\sigma^{-1}\sigma(3, 4)\sigma^{-1}, \]
   so it suffices to find $\sigma$ such that simultaneously,
   \[ \sigma(1, 2)\sigma^{-1} = (5, 6) \]
   and
   \[ \sigma(3, 4)\sigma^{-1} = (3, 1). \]

   By a formula seen in class, it suffices to find $\sigma$ such that $\sigma(1) = 2, \sigma(2) = 6, \sigma(3) = 3$ and $\sigma(4) = 1$. Take
   \[ \sigma = (4, 1, 5)(2, 6) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 5 & 6 & 3 & 1 & 4 & 2 & 7 & 8 & 9 \end{pmatrix}. \]

2. Does there exist $\sigma \in \mathfrak{S}_9$ such that $\sigma(1, 2, 3)\sigma^{-1} = (2, 3)(1, 6, 7)$?

   Solution. Look at signs: the left-hand side is even, the right-hand side is odd, so this is impossible.
3. Does there exist $\sigma \in \mathfrak{S}_9$ such that $\sigma(1, 2, 4)^{-1} = (2, 5)(1, 3)$?

*Solution.* Here both sides are even, so the sign argument does not work. However, by a formula seen in class, we must have

$$\sigma(1, 2, 4)^{-1} = (\sigma(1), \sigma(2), \sigma(3)),$$

that is, the left-hand side is a cycle of length 3, whereas the right-hand side is not, so this is impossible.

**Exercise 4.** The orthogonal group $O_n(\mathbb{R})$ is the subset of $M_n(\mathbb{R})$ given by

$$O_n(\mathbb{R}) = \{ M \in M_n(\mathbb{R}), \ M^tM = MM^t = I_n \}$$

where $M^t$ denotes the transpose of a matrix $M$. We recall that for any matrix $M$, $M$ and $M^t$ have the same determinant.

1. Show that $O_n(\mathbb{R})$ is a subgroup of $(GL_n(\mathbb{R}), \cdot)$.

2. We define the special orthogonal group $SO_n(\mathbb{R})$ to be the subset of $O_n(\mathbb{R})$ of matrices with determinant 1:

$$SO_n(\mathbb{R}) = \{ M \in O_n(\mathbb{R}), \ \det(M) = 1 \}$$

Show that $SO_n(\mathbb{R})$ is a normal subgroup of $O_n(\mathbb{R})$.

3. Show that $SO_n(\mathbb{R})$ has index 2 in $O_n(\mathbb{R})$ and that $O_n(\mathbb{R})/SO_n(\mathbb{R})$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z}, +)$.

*Hint:* Show that the determinant of an element of $O_n(\mathbb{R})$ is either 1 or $-1$.

4. Check that for any real number $\theta$, the matrix

$$M_{\theta} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$$

is an element of $SO_2(\mathbb{R})$.

5. Check that the matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is an element of $O_2(\mathbb{R})$. Is it an element of $SO_2(\mathbb{R})$?

**Exercise 5.** Let $G$ be a group and let $H$ be the *commutator subgroup* of $G$, that is, the set of all finite products of elements of the form $aba^{-1}b^{-1}$ for $a, b \in G$.

1. Show that $H$ is a normal subgroup of $G$.

*Solution.* To check that it is a subgroup, check all the subgroup axioms. To show that it is normal, write for all $g \in G$

$$gaba^{-1}b^{-1}g^{-1} = gag^{-1}gbg^{-1}ga^{-1}gb^{-1}g^{-1} = (gag^{-1})(gbg^{-1})(gag^{-1})^{-1}(gbg^{-1})^{-1}.$$
3. More generally, for any normal subgroup \( N \) of \( G \), show that \( G/N \) is abelian if and only if \( N \) contains \( H \).

**Solution.** By the same method as above, if \( N \) contains \( H \), then \( G/N \) is abelian. Conversely, if \( G/N \) is abelian, then this means that for all \( a, b \in G \), \((aN)(bN) = (bN)(aN)\), that is, \( abN = baN \). Since \( N \) is normal, this implies \( Nab = Nba \) (right cosets are same as left cosets), so \( ab(ba)^{-1} \in N \), i.e. \( aba^{-1}b^{-1} \in N \). Thus, \( N \) contains all of the elements of the form \( aba^{-1}b^{-1} \) for \( a, b \in G \). By closure, it contains all the finite products of such elements, and therefore it contains \( H \).

**Exercise 6.** Let \( \sigma \) be the element of \( S_9 \) given by

\[
\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 8 & 4 & 7 & 9 & 6 & 1 & 3 & 5 & 2 \end{pmatrix}.
\]

1. Give a decomposition of \( \sigma \) into disjoint cycles.

**Solution.** We have \( \sigma = (1, 8, 5, 6)(2, 4, 9)(3, 7) \).

2. Determine the sign of \( \sigma \).

**Solution.** Using multiplicativity of the sign and the fact that a cycle of length \( k \) has sign \((-1)^{k-1}\), we see that is even.

3. What is the order of \( \sigma \) in \( S_9 \)?

**Solution.** Observe that \( \sigma^{12} = \text{id} \) (to compute more quickly, use that disjoint cycles commute, and that a cycle of length \( k \) is of order \( k \)), so that the order of \( \sigma \) divides 12. It is therefore equal to 1, 2, 3, 4, 6 or 12. It cannot be 1 because \( \sigma \neq \text{id} \). We compute

\[
\sigma^2 = (1, 8, 5, 6)^2(2, 4, 9)^2(3, 7)^2 = (1, 5)(8, 6)(2, 9, 4) \neq \text{id}
\]

\[
\sigma^3 = (1, 8, 5, 6)^3(2, 4, 9)^3(3, 7)^3 = (1, 6, 5, 8)(3, 7) \neq \text{id}.
\]

\[
\sigma^4 = (1, 8, 5, 6)^4(2, 4, 9)^4(3, 7)^4 = (2, 4, 9) \neq \text{id}.
\]

\[
\sigma^6 = (1, 8, 5, 6)^6(2, 4, 9)^6(3, 7)^6 = (1, 8, 5, 6)^2 = (1, 5)(8, 6) \neq \text{id}.
\]

Therefore, the order of \( \sigma \) is 12.

**Exercise 7.** In \( S_4 \), consider the subset

\[
H = \left\{ \text{id}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix} \right\}.
\]

1. Compute the inverses of the elements of \( H \) in \( S_4 \).

**Solution.** You should find that \( \text{id}^{-1} = \text{id} \), the first two non-trivial elements are inverse to each other, and the last element is its own inverse.

2. Is \( H \) a subgroup of \( S_4 \)?

**Solution.** No, it does not satisfy closure, looking e.g. at the product of the two last elements.

**Exercise 8.** Let \( n \geq 1 \) be an integer and let \( H = \{ \sigma \in S_n, \sigma(1) = 1 \} \).

1. Show that \( H \) is a subgroup of \( S_n \).
2. Write down all the elements of $H$ when $n = 1$, $n = 2$ and $n = 3$.

3. When $n \geq 3$, show that $H$ is not a normal subgroup of $\mathfrak{S}_n$.

   **Solution.** If $n \geq 3$, then $H$ is not the trivial subgroup $\{\text{id}\}$, and therefore it contains some $\sigma \neq \text{id}$. Then there exists $i > 1$ such that $\sigma(i) \neq i$. By injectivity of $\sigma$, $\sigma(i) \neq \sigma(1) = 1$. Look at the permutation $\alpha = (1, i)\sigma(1, i)^{-1}$: we have
   \[
   \alpha(1) = (1, i)\sigma(i) = \sigma(i) \neq 1,
   \]
   since $\sigma(i) \notin \{1, i\}$. Therefore, $\alpha \notin H$, and so $H$ is not normal.

**Exercise 9.** Let $G$ be a group. Recall that the center of $G$ is the subgroup of $G$ given by
   \[
   Z(G) = \{x \in G, \ xg = gx \text{ for all } g \in G\}.
   \]

1. Show that $Z(G)$ is a normal subgroup of $G$.

2. We assume that the quotient group $G/Z(G)$ is cyclic.

   (a) Show that this implies the existence of some element $t \in G$ such that for all $a \in G$, the coset $aZ(G)$ is equal to $t^nZ(G)$ for some $n \in \mathbb{Z}$.

   **Solution.** Since $G/Z(G)$ is cyclic, it is generated by some coset $tZ(G)$ for some $t \in G$. This means that for all $aZ(G)$, there is $n \in \mathbb{Z}$ such that $aZ(G) = (tZ(G))^n = t^nZ(G)$, where the last equality comes from the definition of the group law in $G/Z(G)$.

   (b) Show that if $aZ(G) = t^nZ(G)$, then there exists $x \in Z(G)$ such that $a = t^n x$. **Solution.** Two elements $a$ and $b$ define the same coset if and only if $b^{-1}a \in Z(G)$, so if and only if $a = bx$ for some $x \in Z(G)$. Apply this to $b = t^n$.

   (c) Deduce from this that $G$ is abelian.

   **Solution.** Let $a, b \in G$. We want to prove that $ab = ba$. Using the previous question, we may write $a = t^n x$ and $b = t^m y$ for $m, n \in \mathbb{Z}$ and $x, y \in Z(G)$. Then we have
   \[
   ab = t^n xt^m y \\
   = t^{n+m} xy \quad \text{because } x \in Z(G) \\
   = t^{m+n} xy \\
   = t^m t^n xy \quad \text{because } x \in Z(G) \\
   = t^m y t^n x \quad \text{because } y \in Z(G) \\
   = ba
   \]

**Exercise 10.** Let $G$ be a group and let $H$ be a subgroup of $G$. Recall that for all $g \in G$, $gHg^{-1}$ is a subgroup of $G$. We define $N$ to be the intersection of all these subgroups.

1. Show that it is a normal subgroup of $G$.

2. Show that if $H$ is normal, then $H = N$.

   **Solution.** If $H$ is normal, then for all $g \in G$, $gHg^{-1} = H$, so the intersection of all of these subgroups is $H$.

3. Compute $N$ when $G = \mathfrak{S}_3$ and $H = \{\text{id}, (12)\}$.

   **Solution.** Compute e.g. $(12)H(12)^{-1} = H$ and $(13)H(13)^{-1} = \{\text{id}, (23)\}$. Already the intersection of these two subgroups is trivial, so the total intersection will be trivial as well.