Algebra practice problems

Exercise 1. Let $G$ be a group and let $H_1, H_2$ be normal subgroups of $G$. Show that $H_1 \cap H_2$ is a normal subgroup of $G$.

Exercise 2. Let $H$ be a subgroup of a group $G$. The centralizer of $H$ in $G$ is defined to be the set

$$C_H(G) = \{ x \in G, \ xh = hx \text{ for all } h \in H \}.$$

1. Show that $C_H(G)$ is a subgroup of $G$.
2. Show that if $H$ is normal, then $C_H(G)$ is normal.

Exercise 3. 1. Find a permutation $\sigma \in S_9$ such that $\sigma(1,2)(3,4)\sigma^{-1} = (5,6)(3,1)$.

2. Does there exist $\sigma \in S_9$ such that $\sigma(1,2,3)\sigma^{-1} = (2,3)(1,6,7)$?

3. Does there exist $\sigma \in S_9$ such that $\sigma(1,2,4)\sigma^{-1} = (2,5)(1,3)$?

Exercise 4. The orthogonal group $O_n(\mathbb{R})$ is the subset of $M_n(\mathbb{R})$ given by

$$O_n(\mathbb{R}) = \{ M \in M_n(\mathbb{R}), \ M^tM = MM^t = I_n \}$$

where $M^t$ denotes the transpose of a matrix $M$. We recall that for any matrix $M$, $M$ and $M^t$ have the same determinant.

1. Show that $O_n(\mathbb{R})$ is a subgroup of $(GL_n(\mathbb{R}), \cdot)$.
2. We define the special orthogonal group $SO_n(\mathbb{R})$ to be the subset of $O_n(\mathbb{R})$ of matrices with determinant 1:

$$SO_n(\mathbb{R}) = \{ M \in O_n(\mathbb{R}), \ det(M) = 1 \}.$$

Show that $SO_n(\mathbb{R})$ is a normal subgroup of $O_n(\mathbb{R})$.

3. Show that $SO_n(\mathbb{R})$ has index 2 in $O_n(\mathbb{R})$ and that $O_n(\mathbb{R})/SO_n(\mathbb{R})$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z}, +)$.

4. Check that for any real number $\theta$, the matrix

$$M_{\theta} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$$

is an element of $SO_2(\mathbb{R})$.

5. Check that the matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is an element of $O_2(\mathbb{R})$. Is it an element of $SO_2(\mathbb{R})$?

Exercise 5. Let $G$ be a group and let $H$ be the commutator subgroup of $G$, that is, the set of all finite products of elements of the form $aba^{-1}b^{-1}$ for $a, b \in G$.

1. Show that $H$ is a normal subgroup of $G$.
2. Show that the quotient $G/H$ is abelian.
3. More generally, for any normal subgroup $N$ of $G$, show that $G/N$ is abelian if and only if $N$ contains $H$. 
Exercise 6. Let $\sigma$ be the element of $S_9$ given by

$$\sigma = \left( \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 8 & 4 & 7 & 9 & 6 & 1 & 3 & 5 & 2 \end{array} \right).$$

1. Give a decomposition of $\sigma$ into disjoint cycles.
2. Determine the sign of $\sigma$.
3. What is the order of $\sigma$ in $S_9$?

Exercise 7. In $S_4$, consider the subset

$$H = \{ \text{id}, \left( \begin{array}{ccc} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 4 & 4 & 4 \end{array} \right), \left( \begin{array}{ccc} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 4 & 4 & 4 \end{array} \right), \left( \begin{array}{ccc} 1 & 2 & 3 \\ 1 & 4 & 3 \\ 2 & 4 & 4 \end{array} \right) \}.$$

1. Compute the inverses of the elements of $H$ in $S_4$.
2. Is $H$ a subgroup of $S_4$?

Exercise 8. Let $n \geq 1$ be an integer and let $H = \{ \sigma \in S_n, \sigma(1) = 1 \}$.

1. Show that $H$ is a subgroup of $S_n$.
2. Write down all the elements of $H$ when $n = 1$, $n = 2$ and $n = 3$.
3. When $n \geq 3$, show that $H$ is not a normal subgroup of $S_n$.

Exercise 9. Let $G$ be a group. Recall that the center of $G$ is the subgroup of $G$ given by

$$Z(G) = \{ x \in G, \; xg = gx \text{ for all } g \in G \}.$$

1. Show that $Z(G)$ is a normal subgroup of $G$.
2. We assume that the quotient group $G/Z(G)$ is cyclic.
   (a) Show that this implies the existence of some element $t \in G$ such that for all $a \in G$, the coset $aZ(G)$ is equal to $t^nZ(G)$ for some $n \in \mathbb{Z}$.
   (b) Show that if $aZ(G) = t^nZ(G)$, then there exists $x \in Z(G)$ such that $a = t^n x$.
   (c) Deduce from this that $G$ is abelian.

Exercise 10. Let $G$ be a group and let $H$ be a subgroup of $G$. Recall that for all $g \in G$, $gHg^{-1}$ is a subgroup of $G$. We define $N$ to be the intersection of all these subgroups.

1. Show that it is a normal subgroup of $G$.
2. Show that if $H$ is normal, then $H = N$.
3. Compute $N$ when $G = S_3$ and $H = \{ \text{id}, (12) \}$.