Midterm
October 31, 2017

The use of notes and electronic devices is forbidden. All answers have to be justified. You can use all results proved in lectures without proof. If you use a result seen in a homework or in recitation you need to prove it.

Exercise 1. 1. Let $f : G \to H$ be a group homomorphism. Define $\text{Ker } f$ and show that it is a subgroup of $G$.

2. Let $A$ be the subset of $M_2(\mathbb{R})$ given by

$$A = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{R}), \ a + d = 0 \right\}.$$ 

Is it a subgroup of $(M_2(\mathbb{R}), +)$?

Exercise 2. We denote the class of an integer $n$ in $\mathbb{Z}/10\mathbb{Z}$ by $[n]$.

1. (a) Give the order of $[2]$ in the group $(\mathbb{Z}/10\mathbb{Z}, +)$. Is it a generator?
   
   (b) Show that the group $\mathbb{Z}/10\mathbb{Z}$ is cyclic and give the list of all its generators.
   
   (c) Give an element of order 2 in $\mathbb{Z}/10\mathbb{Z}$.
   
   (d) Give a subgroup of order 5 of $\mathbb{Z}/10\mathbb{Z}$.

2. Let $\phi : \mathbb{Z}/10\mathbb{Z} \to \mathbb{Z}/10\mathbb{Z}$ be a group homomorphism.

   (a) Can we have $\phi([0]) = [3]$?
   
   (b) If $\phi([2]) = [4]$, what is $\phi([8])$?
   
   (c) If $\phi([1]) = [2]$, can $\phi$ be an isomorphism?
   
   (d) Show that $\phi$ is of the form $[n] \mapsto a[n]$ for some $a \in \mathbb{Z}/10\mathbb{Z}$.
   
   (e) Check that any map of this type is indeed a homomorphism.
   
   (f) For which values of $a$ is the homomorphism $[n] \mapsto a[n]$ an isomorphism?

3. (a) What is the order of the group $((\mathbb{Z}/10\mathbb{Z})^\times, \cdot)$? Give all its elements and compute their inverses.
   
   (b) Describe the group $(\mathbb{Z}/10\mathbb{Z})^\times$ by giving its Cayley table.
   
   (c) Is $(\mathbb{Z}/10\mathbb{Z})^\times$ cyclic? If yes, give a generator.
   
   (d) Is there an element of order 3 in $(\mathbb{Z}/10\mathbb{Z})^\times$?
   
   (e) Give an example of a proper subgroup of $(\mathbb{Z}/10\mathbb{Z})^\times$.

Exercise 3. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be the map defined by $(x, y) \mapsto (x + y, y)$.

1. Show that $f$ is a group homomorphism.

2. Determine its kernel and its image.

3. Is $f$ an isomorphism?

Exercise 4. 1. Give a list of all groups of order at most 4 up to isomorphism.

2. Let $G$ be a group of order 4 having at least two distinct elements of order 2. Determine which of the groups in the list from the previous question it is isomorphic to.

3. Give a list of all the subgroups of this group.