Algebra homework 6
Homomorphisms, isomorphisms

Exercise 1. Show that the following maps are group homomorphisms and compute their kernels.

(a) $f : (\mathbb{R}^\times, \cdot) \to (GL_2(\mathbb{R}), \cdot)$ given by
$$f(x) = \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}.$$ 

(b) $g : (\mathbb{R}, +) \to (GL_2(\mathbb{R}), \cdot)$ given by
$$g(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}.$$ 

(c) $h : (\mathbb{R}^2, +) \to (\mathbb{R}, +)$ given by $h(x, y) = x$.

(d) The complex conjugation map $j : (\mathbb{C}, +) \to (\mathbb{C}, +)$, given by $j(x + iy) = x - iy$.

(e) $k : G \to G$ given by $k(x) = x^n$ if $G$ is an abelian group (written in multiplicative notation).
What if $G$ is not abelian?

Exercise 2. Let $\phi : G \to H$ be a group homomorphism.

1. Show that if $G$ is abelian, then $\text{Im}(\phi)$ is also abelian.

2. Show that if $G$ is cyclic, then $\text{Im}(\phi)$ is also cyclic.

Exercise 3. Let $T$ denote the group of invertible upper triangular $2 \times 2$ matrices
$$A = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}, \quad a, b, d \in \mathbb{R}, \ ad \neq 0.$$ 

1. Show that $T$ is a subgroup of $GL_2(\mathbb{R})$.

2. Let $\phi : T \to \mathbb{R}^\times$ be the map given by sending a matrix $A$ as above to $a^2$. Show that $\phi$ is a homomorphism, and give its kernel and image.

Exercise 4. Show that all the non-trivial subgroups of $\mathbb{Z}$ are isomorphic to $\mathbb{Z}$.

Exercise 5. Recall that the group $(\mathbb{Z}/8\mathbb{Z})^\times$ is of order 4. Is it isomorphic to $\mathbb{Z}/4\mathbb{Z}$? If not, find another group of order 4 it is isomorphic to.

Exercise 6. 1. Show that for any $a \in \mathbb{Z}$, the map $\phi : \mathbb{Z} \to \mathbb{Z}$ defined by $\phi(n) = an$ is a group homomorphism. Give its kernel and image.

2. Conversely, show that a homomorphism $\phi : \mathbb{Z} \to \mathbb{Z}$ is of the form $\phi(n) = an$ for some $a \in \mathbb{Z}$. Thus, the homomorphisms $\mathbb{Z} \to \mathbb{Z}$ are exactly the maps $n \mapsto an$.

3. Determine all the automorphisms of $\mathbb{Z}$, that is, the isomorphisms $\mathbb{Z} \to \mathbb{Z}$.

Exercise 7. Let $G$ and $H$ be two groups. Show that $G \times H$ is isomorphic to $H \times G$.

Exercise 8. Are the groups $\mathbb{Z}/6\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ isomorphic? Justify your answer by either constructing an isomorphism or explaining why it does not exist.