Algebra homework 2
Arithmetic on the set of integers

Notation: For a set $A$ with a finite number of elements, we denote by $|A|$ the number of its elements.

**Exercise 1.** Prove the following properties:

(a) For every integer $a$, the integers $1$, $-1$, $a$ and $-a$ divide $a$.
(b) $0$ does not divide any non-zero integer.
(c) All integers divide $0$.
(d) If $a, b, c$ are integers such that $a|b$ and $b|c$ then $a|c$.
(e) If $a, b$ are non-zero integers, then $a|b$ and $b|a$ implies $a = b$ or $a = -b$.

**Exercise 2.** Give the list of the positive divisors of the following numbers:

(a) $20$
(b) $57$

**Exercise 3.** Let $p$ be a prime number, that is, a positive number such that its only positive divisors are $1$ and $p$. Give the list of all the positive divisors of $p^2$, then of $p^3$. More generally, describe, in terms of $p$ and $k$, the list of positive divisors of $p^k$ for any integer $k \geq 1$.

**Exercise 4.** Let $n$ be a positive integer. Show that the smallest integer $d > 1$ dividing $n$ is a prime number.

**Exercise 5.** Let $a, b$ be two integers, not both zero. Recall that in the lectures, their greatest common divisor $\gcd(a, b)$ has been defined as the largest positive number $d$ which divides both $a$ and $b$. Show that any non-zero integer $e$ dividing both $a$ and $b$ divides $\gcd(a, b)$.

**Exercise 6.** For any integers $a, b$ which are not both zero, prove the following properties of the greatest common divisor:

(a) $\gcd(a, b) = \gcd(b, a)$.

(b) For any integer $k \geq 1$, $\gcd(ka, kb) = k \gcd(a, b)$.

(c) If $d = \gcd(a, b)$, then there exist relatively prime integers $a', b'$ such that $a = da'$ and $b = db'$.

(d) $\gcd(a, b) = \gcd(a + b, b)$. 

(e) $\gcd(a, a + 1) = 1$

(f) For any integer $k \geq 1$, $\gcd(a, a + k)$ divides $k$.

**Exercise 7.** Find $\gcd(231, 163)$, as well as integers $u, v$ such that

$$231u + 163v = \gcd(231, 163).$$

**Exercise 8.** The Euler function $\phi : \mathbb{N} \rightarrow \mathbb{N}$ is the function defined for every positive integer $n$ by

$$\phi(n) = |\{k \in \{1, \ldots, n\}, \ k \text{ relatively prime to } n\}|.$$

1. What is the value of $\phi(p)$ for a prime number $p$?

2. What is, in terms of $k$, the value of $\phi(p^k)$ where $p$ is a prime number and $k \geq 1$ an integer?

3. Compute $\phi(n)$ for all integers $n$ in the set \{1, 2, \ldots, 12\}.

**Exercise 9.** Let $n$ be an integer, and $a, b$ non-zero relatively prime integers. Show that if both $a$ and $b$ divide $n$, then the product $ab$ divides $n$. Does this remain true if $a$ and $b$ are no longer assumed to be relatively prime?