Theory of Numbers practice problems

Exercise 1. Solve the following Diophantine equations:

(a) $62x + 34y = 2$
(b) $15x + 33y = 7$
(c) $19x + 99y = 3$.

Exercise 2. Use the Euclidean algorithm (by hand, no calculators!) to compute $\gcd(54321, 9876)$.

Exercise 3. Find the remainders of $2^{50}$ and $41^{65}$ in the Euclidean division by 7.

Exercise 4. Solve the following linear congruences:

(a) $7x \equiv 13 \pmod{100}$
(b) $15x \equiv 12 \pmod{57}$
(c) $15x \equiv 12 \pmod{53}$
(d) $23x \equiv 5 \pmod{91}$

Exercise 5. Prove that for any positive integer $n$, the following congruences hold:

(a) $2^{2n} \equiv 1 \pmod{3}$.
(b) $2^{3n} \equiv 1 \pmod{7}$.
(c) $2^{4n} \equiv 1 \pmod{15}$.

Exercise 6. Solve each of the following sets of simultaneous congruences:

(a) $x \equiv 1 \pmod{3}, \ x \equiv 2 \pmod{5}, \ x \equiv 3 \pmod{7}$;
(b) $2x \equiv 1 \pmod{5}, \ 3x \equiv 9 \pmod{6}, \ 4x \equiv 1 \pmod{7}, \ 5x \equiv 9 \pmod{11}$.

Exercise 7. For positive integers $m$ and $n$ prove that $\phi(m)\phi(n) = \phi(mn)\phi(d)/d$ where $d = \gcd(m, n)$.

Exercise 8. If $p$ is an odd prime, prove that the congruence

$$x^{p-2} + \ldots + x^2 + x + 1 \equiv 0 \pmod{p}$$

has exactly $p - 2$ incongruent solutions and they are $2, 3, \ldots, p - 1$.

Exercise 9. Determine the following:

1. $v_2(100^k)$ for $k \geq 1$.
2. $v_5(5555)$.
3. $v_3(1234567)$.
4. $v_5(3 + 3^2 + 3^3 + \ldots + 3^{999})$.
5. $v_{37}(37!)$.
6. $v_2(3^k + 1)$ for $k \geq 0$. 