Theory of Numbers homework 8
Number-theoretic functions
Due November 12th, 2018

Please hand in your homework stapled, with your name written on it. All answers have to be justified, and details of calculations must be given.

In what follows, \( \tau(n) \) denotes the number of positive divisors of \( n \), \( \sigma(n) \) the sum of the positive divisors of \( n \), and \( \mu \) is the Möbius function.

**Exercise 1.** Prove that \( \sigma(n) \) is an odd integer if and only if \( n \) is a perfect square or twice a perfect square.

**Solution.** Express \( n = 2^k \prod_i p_i^{\alpha_i} \) where the \( p_i \) are odd distinct primes. Then

\[
\sigma(n) = \frac{2^{k+1} - 1}{2 - 1} \prod_i \frac{p_i^{\alpha_i+1} - 1}{p_i - 1} = (2^{k+1} - 1) \prod_i \frac{p_i^{\alpha_i+1} - 1}{p_i - 1}
\]

\( \sigma(n) \) is odd if and only if each of the terms multiplied above is odd. We know that \( 2^{k+1} - 1 \) is always odd, so \( k \) can have any value. Now, modulo 2, for \( p_i \neq 2 \),

\[
\frac{p_i^{\alpha_i+1} - 1}{p_i - 1} = \sum_{\beta=0}^{\alpha_i} p_i^{\beta} \equiv \begin{cases} 1 & \alpha_i \text{ even} \\ 0 & \alpha_i \text{ odd} \end{cases} \tag{1}
\]

In particular, this shows that each of the \( \alpha_i \) must be even. Recall that \( k \) is arbitrary. Thus, if \( k \) is odd, we have that \( n \) is twice a perfect square. If \( k \) is even, \( n \) is a perfect square.

Suppose conversely that \( n \) is twice a perfect square. Then, a direct calculation shows that \( \sigma(n) \) is odd, because each of the terms like (1) above are odd.

**Exercise 2.** Show that \( \sum_{d|n} \frac{1}{d} = \frac{\sigma(n)}{n} \) for every positive integer \( n \)

**Solution.** We know from the class notes that

\[
\sigma(n) = \sum_{d|n} d.
\]

This implies that

\[
\frac{\sigma(n)}{n} = \sum_{d|n} \frac{1}{d} = \sum_{d|n} \frac{1}{d}.
\]

The second equality follows from the fact that every \( \frac{1}{d} \) is of the form \( \frac{k}{n} \) for \( kd = n \).

**Exercise 3.** 1. Describe the integers \( n > 1 \) which satisfy \( \tau(n) = 6 \).

   **Solution.** According to the formula for \( \tau \), these are the integers of the form \( p^5 \) for some prime number \( p \), or \( pq^2 \) for distinct prime numbers \( p \) and \( q \).

2. Determine the integers \( n > 1 \) such that \( \sigma(n) = 6 \).

   **Solution.** Recall that \( \sigma(n) \geq n+1 \) for all \( n > 1 \), so that we only need to check the integers \( n \leq 5 \). We find that the only solution is \( n = 5 \).
3. Given a positive integer \( m > 1 \), show that there are infinitely many integers \( n \) for which \( \tau(n) = m \), but at most finitely many with \( \sigma(n) = m \).

*Solution.* Let \( m > 1 \) be given. Take \( p \) a prime. Then \( \tau(p^{m-1}) = m \). This is because the divisors of \( p^{m-1} \) are \( 1, p, p^2, \ldots, p^{m-1} \). There are infinitely many primes, so infinitely many \( n \in \mathbb{N} \) satisfy \( \tau(n) = m \).

On the other hand, we know that \( \sigma(n) \geq n + 1 \). So, for \( n \geq m \), we see \( \sigma(n) \geq m + 1 > m \). So, if \( \sigma(n) = m \), \( n \leq m - 1 \). Therefore, there are at most finitely many solutions to \( \sigma(n) = m \).

**Exercise 4.** Given \( n \geq 1 \), let \( \sigma_s(n) \) denote the sum of the \( s \)-th powers of the positive divisors of \( n \):

\[
\sigma_s(n) = \sum_{d|n} d^s.
\]

1. Check that \( \tau = \sigma_0 \) and \( \sigma = \sigma_1 \).

*Solution.* This is by definition.

\[
\sigma_0(n) = \sum_{d|n} d^0 = \sum_{d|n} 1.
\]

That is, we sum up 1’s indexed by the divisors of \( n \). So, this counts the number of divisors of \( n \). So, \( \tau(n) = \sigma_0(n) \). Next,

\[
\sigma_1(n) = \sum_{d|n} d^1 = \sum_{d|n} d = \sigma(n).
\]

2. Show that \( \sigma_s \) is a multiplicative function.

*Solution.*

Note that \( n \mapsto n^s \) is a multiplicative function. The result then follows from proposition 7.2.4 in the lecture notes.

3. If \( n = p_1^{\alpha_1} \cdots p_r^{\alpha_r} \) is the prime factorization of \( n \), show that

\[
\sigma_s(n) = \left( \frac{p_1^{s(\alpha_1+1)} - 1}{p_1^s - 1} \right) \left( \frac{p_2^{s(\alpha_2+1)} - 1}{p_2^s - 1} \right) \cdots \left( \frac{p_r^{s(\alpha_r+1)} - 1}{p_r^s - 1} \right).
\]

*Solution.* This follows at once from the decomposition of the \( n \)-fold summation above, coupled with the fact that

\[
\sum_{j_i = 0}^{\alpha_i} p_i^{sj_i} = \frac{p_i^{s(\alpha_i+1)} - 1}{p_i^s - 1}
\]

using the geometric sum formula.

**Exercise 5.** Let \( f \) and \( g \) be two multiplicative number-theoretic functions. Assume they both are not identically zero (meaning that each of them takes at least one non-zero value), and that \( f(p^k) = g(p^k) \) for each prime \( p \) and each integer \( k \geq 1 \). Show that \( f(n) = g(n) \) for all positive integers \( n \).
Solution. We can write a positive integer $n \in \mathbb{N}$ as $n = \prod_i p_i^{\alpha_i}$ where the $p_i$ are distinct primes. By multiplicativity, we have

$$f(n) = \prod_i f(p_i^{\alpha_i}) = \prod_i g(p_i^{\alpha_i}) = g(n).$$

In particular, $f$ and $g$ agree for all $n \in \mathbb{N}$.

Exercise 6. Let $\omega(n)$ denote the number of distinct prime divisors of $n > 1$, with $\omega(1) = 0$.

1. Compute $\omega(600)$.

Solution. Factorize as $600 = 2^3 \times 3 \times 5^2$. Then, there are three distinct prime divisors with $n > 1$. So, $\omega(600) = 3$.

2. Show that the function $f$ defined by $f(n) = 2^{\omega(n)}$ is multiplicative.

Solution. Suppose that $m, n \in \mathbb{N}$ are relatively prime. Let $q_1, \ldots, q_k$ denote the positive prime divisors of $m$ and $p_1, \ldots, p_l$ those of $n$. By coprimality, the $q_i$ and $p_j$ are distinct. Now, a prime $r$ divides $mn$ if and only if it divides one of $m$ or $n$. So, there is a bijection between prime divisors of $m$ or $n$ and prime divisors of $mn$. In particular, $\omega(n) + \omega(m) = \omega(mn)$. So, $2^{\omega(n)} \cdot 2^{\omega(m)} = 2^{\omega(n) + \omega(m)}$. So, $f$ is multiplicative:

$$f(n)f(m) = 2^{\omega(n)} \cdot 2^{\omega(m)} = 2^{\omega(n) + \omega(m)} = f(mn).$$

3. For any positive $n$, establish the formula

$$\tau(n^2) = \sum_{d|n} 2^{\omega(d)}.$$

Solution. Both sides of the equation are multiplicative functions (recall that we proved in lectures that if $f$ is multiplicative, then $n \mapsto \sum_{d|n} f(d)$ is multiplicative), so by exercise 5, it suffices to check this when $n$ is of the form $p^k$ for a prime $p$ and an integers $k \geq 1$. We have $\tau(n^2) = \tau(p^{2k}) = 2k + 1$. On the other hand, the divisors of $n = p^k$ are $1, p, p^2, \ldots, p^k$, and we have $\omega(1) = 0$, but $\omega(p^a) = 1$ for $a = 1, 2, \ldots, k$, so that

$$\sum_{d|p^k} 2^{\omega(d)} = 2^0 + k \times 2^1 = 1 + 2k,$$

whence the result.

Exercise 7. For every positive integer $n$, show that

$$\mu(n)\mu(n + 1)\mu(n + 2)\mu(n + 3) = 0.$$

Solution. The 4 consecutive integers $n, n + 1, n + 2, n + 3$ have different remainders modulo 4, so one of them is divisible by 4, and therefore not square-free, so that the value of $\mu$ at this integer is zero. Therefore the above product is zero as well.

Exercise 8. For every integer $n \geq 3$, show that

$$\sum_{k=1}^{n} \mu(k!) = 1.$$

Solution. For any $k \geq 4$, $k!$ is divisible by 4, so not square-free, so $\mu(k!) = 0$. Therefore the above sum is equal to

$$\mu(1!) + \mu(2!) + \mu(3!) = \mu(1) + \mu(2) + \mu(6) = 1 - 1 + 1 = 1.$$