Theory of Numbers homework 3  
Prime numbers, Pythagorean triples  
Due October 1st, 2018

Please hand in your homework stapled, with your name written on it. All answers have to be justified.

**Exercise 1.** Show that any composite three-digit number must have a prime factor less than or equal to 31.

*Solution.* Let \( n \) be such a number. In particular, \( n \leq 999 \). Let \( n = \prod p_i^{\alpha_i} \), a prime factorization. If \( \min\{p_i\} > 31 \), then we know that \( n \geq 37^2 \), because \( n \) is composite, and 37 is the next smallest prime. But then \( n \geq 37^2 = 1369 > 999 \), so this is impossible. Hence, \( \min\{p_i\} \leq 31 \).

**Exercise 2.** A positive integer \( n \) is called square-full if \( p^2 \) divides \( n \) for every prime factor \( p \) of \( n \). If \( n \) is square-full, show that it can be written in the form \( n = a^2b^3 \) with \( a \) and \( b \) positive integers.

*Solution.* We can write the prime factorization \( n = \prod p_i^{\alpha_i} = \prod p_i^{v_{p_i}(n)} \). By assumption \( v_{p_i}(n) \geq 2 \) for all \( i \). Every number \( \geq 2 \) can be written uniquely as \( 2^k \) or \( 2^k + 3 \) with non-negative \( k \), according to whether it is even or not. For each \( p_i \mid n \) write \( v_{p_i}(n) = 2k_i + 3b_i \) for \( b \in \{0, 1\} \). For all \( i \) write \( p_i^{v_{p_i}(n)} = p_i^{2k_i} p_i^{3b_i} \). Then
\[
\prod p_i^{v_{p_i}(n)} = \left( \prod p_i^{2k_i} \right) \left( \prod p_i^{3b_i} \right)
\]
This is the requisite decomposition.

**Exercise 3.** Find all (not necessarily primitive) Pythagorean triples \((x, y, z)\) such that \( x = 15 \) and \( x, y, z > 0 \).

*Solution.* By Remark 4.1.4 of the class notes, we know that all Pythagorean triples are obtained by rescaling primitive triples. There are several cases to consider, according to the value of \( \gcd(x, y, z) \).

If \( \gcd(x, y, z) = 1 \), then the triple is primitive with \( x \) odd, so by Theorem 4.2.2 in the notes, solutions are given by the formulae
\[
x = st, \quad y = \frac{s^2 - t^2}{2}, \quad z = \frac{s^2 + t^2}{2}
\]
for all possible odd and relatively prime values of \( s, t \) with \( s > t > 0 \). Since the positive divisors of \( x = 15 \) are 1, 3, 5, 15, we have only two possible triples: for \( s = 5 \) and \( t = 3 \) we get \((15, 8, 17)\) and for \( s = 15 \) and \( t = 1 \), we have \((15, 112, 113)\). This exhausts the primitive triples with \( x = 15 \).

Now, we turn to the non-primitive triples. In this case, by remark 4.1.4 in the notes, they must arise as rescalings of primitive triples \((a, b, c)\) with \( a|15 : a = 3 \) if \( \gcd(x, y, z) = 5 \), and \( a = 5 \) if \( \gcd(x, y, z) = 3 \) (\( a = 1 \) yields no triples by Exercise 4). Since the numbers 3 and 5 are prime, there is only one set of values for \( s \) and \( t \) in both cases, and we get unique Pythagorean triples
(3, 4, 5) and (5, 12, 13). Rescaling these by the respective values of the gcd, we have triples (15, 20, 25) and (15, 36, 39) respectively. So, all of the Pythagorean triples with \( x = 15 \) are
\[
(15, 8, 17), (15, 112, 113), (15, 20, 25), (15, 36, 39).
\]

Exercise 4. 1. Let \((x, y, z)\) be a Pythagorean triple. Show that none of the integers \(x, y, z\) can be equal to 1 or 2.

Solution. Suppose that \(z \in \{1, 2\}\). \(z = 1\) is impossible because \(x^2 + y^2 \geq 2\). If \(z = 2\), note that \(x^2 + y^2\) has a sequence of values in increasing order given by 2, 5, 8, 10, \ldots So, this case is also impossible. Now, we’re in the case where without loss of generality \(y \in \{1, 2\}\). Write
\[
y^2 = z^2 - x^2 = (z - x)(z + x).
\]
Without loss of generality, we may assume \(x, z > 0\). If \(y = 1\), then, since \(z + x > 0\), we must have \(z - x = 1\) and \(z + x = 1\). Adding up these, we get \(z = 1\), a contradiction If \(y = 2\), then we have, \(4 = (z - x)(z + x)\). Since \(z - x < z + x\), we must have \(z - x = 1\) and \(z + x = 4\), which is impossible because adding these equations up, we see that \(z\) cannot be an integer.

2. Let \(n \geq 3\) be an integer. Show that there exists a Pythagorean triple \((x, y, z)\) with \(x, y, z > 0\) such that \(n\) is equal to one of the integers \(x, y\) or \(z\).

Solution. If \(n\) is odd, we may use the formula for primitive Pythagorean triples with \(s = n\) and \(t = 1\) to get the triple \((n, \frac{n^2-1}{2}, \frac{n^2+1}{2})\). If \(n\) is even, denoting \(k = v_2(n)\), we may write \(n = 2^k n'\) where \(n'\) is odd. Multiplying the primitive triple \((n', \frac{n'^2-1}{2}, \frac{n'^2+1}{2})\), obtained as in the previous case, by \(2^k\), we get the triple
\[
(n, 2^k n'^2 - 1, 2^k n'^2 + 1).
\]

Exercise 5. 1. Prove that \((3, 4, 5)\) is the only Pythagorean triple involving three consecutive positive integers.

2. Check that \((3n, 4n, 5n)\) is a Pythagorean triple for all integers \(n\).

3. Recall that three integers \(x, y, z\) are said to be in arithmetic progression if \(y - z = z - y\). Show that if \((x, y, z)\) is a Pythagorean triple such that \(x, y, z\) are in arithmetic progression, then \((x, y, z) = (3n, 4n, 5n)\) for some integer \(n\).

Solution. 1. Suppose such a triple \((n - 1, n, n + 1)\) is given. If the equation is to be solved, then we must write
\[
(n - 1)^2 + n^2 = (n + 1)^2.
\]
After some arithmetic, we have \(n^2 - 4n = 0\). So, \(n(n - 4) = 0\). Thus, \(n = 0\) or \(n = 4\). Tossing out the edge case we have \(n = 4\). Thus, the triple was 3, 4, 5 after all.

2. Given that \(3^2 + 4^2 = 5^2\), multiply both sides by \(n^2\) to see \((3n)^2 + (4n)^2 = (5n)^2\).

3. This is basically the same as part 1. Any such triple can be written as \((n - k, n, n + k)\). Then again, the only way for the equality to hold is
\[
(n - k)^2 + n^2 = (n + k)^2.
\]
Expanding terms and simplifying, one has \(n(n - 4k) = 0\). So, \(n = 4k\) because \(n \neq 0\). Then our sequence reads \((3k, 4k, 5k)\). So, we are done.