Introduction to Number Theory homework 7
Prime number theorem and further topics

Hand in your homework stapled, with your name written on it. All answers have to be justified.

Exercise 1. 1. (Abel summation) Let \((a_n)_{n \geq 1}\) be a sequence of complex numbers, and for any \(t\), put

\[ A(t) = \sum_{1 \leq n \leq t} a_n. \]

For any real numbers \(x < y\) and any continuously differentiable function \(f\) on the interval \([x, y]\), show that we have

\[ \sum_{x < n \leq y} a_n f(n) = A(y)f(y) - A(x)f(x) - \int_x^y A(u)f'(u)du. \]

2. Let \(S(x) = \sum_{1 \leq n \leq x} \frac{\Lambda(n)}{\log(n)}\). Show that

\[ S(x) = \pi(x) + O(\sqrt{x}\log(x)), \]

where \(\pi(x)\) is the prime counting function.

3. Assume that we know that there exists a positive constant \(c\) such that

\[ \psi(x) = x + O(xe^{-c\sqrt{\log(x)}}). \]

Show that then there exists a positive constant \(c'\) such that

\[ \pi(x) = \int_2^x \frac{du}{\log(u)} + O(xe^{-c'\sqrt{\log(x)}}). \]

Exercise 2. 1. The Bernoulli numbers \((B_j)_{j \geq 0}\) are given by the generating function

\[ \frac{z}{e^z - 1} = \sum_{j \geq 0} B_j \frac{z^j}{j!}. \]

Show the recursive formula

\[ \sum_{k=0}^{m} \binom{m+1}{k} B_k = \delta_{m,0} \]

where \(\delta_{m,0} = 1\) if \(m = 0\) and 0 otherwise.

2. We define \(S_m(n) = \sum_{k=1}^{n} k^m\). Show that

\[ S_m(n) = \frac{1}{m+1} \sum_{j=0}^{m} (-1)^j \binom{m+1}{j} B_j n^{m+1-j}. \]