Introduction to Number Theory homework 1
Introduction, Riemann zeta function

Hand in your homework stapled, with your name written on it. All answers have to be justified. In all what follows, log denotes the natural logarithm, and $p$ always denotes a prime number.

Exercise 1. The von Mangoldt function is defined on the positive integers by

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^k \text{ for some prime } p \text{ and some } k \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

and the two Chebychev functions on positive reals respectively by

$$\theta(x) = \sum_{p \leq x} \log(p), \quad \text{and} \quad \psi(x) = \sum_{0 < n \leq x} \Lambda(n).$$

1. For any integer $n \geq 1$, show that

$$\sum_{d|n} \Lambda(d) = \log(n).$$

2. For any positive real number $x$, show that

$$\psi(x) = \sum_{m \geq 1} \theta(x^{\frac{1}{m}}).$$

3. For any integer $m \geq 1$, show that

$$e^{\psi(m)} = \text{lcm}(1, 2, \ldots, m).$$

Exercise 2.

1. (Euler’s summation formula) Let $f$ be a continuously differentiable function on an interval $[x, y]$, where $0 < x < y$. Prove that we have the following:

$$\sum_{x < n \leq y} f(n) = \int_x^y f(t)dt + \int_x^y (t - [t])f'(t)dt + f(y)(y - y) - f(x)(x - x)$$

where $[z]$ denotes the integer part of $z$, i.e. the largest integer $m$ such that $m \leq z$.

2. Show that the limit

$$\lim_{x \to \infty} \left( \sum_{n \leq x} \frac{1}{n} - \log x \right)$$

exists. It is denoted $\gamma$ and called the Euler-Mascheroni constant. Check moreover that, as $x$ goes to infinity, we have

$$\sum_{n \leq x} \frac{1}{n} = \log x + \gamma + O \left( \frac{1}{x} \right).$$
3. Verify that
\[
\lim_{s \to 1} \left( \zeta(s) - \frac{1}{s-1} \right) = \gamma.
\]

**Exercise 3.** We denote by \(\pi(x)\) the number of primes \(p\) such that \(1 < p \leq x\). The aim of this exercise is to adapt the proof of Euclid’s theorem of the infiniteness of primes to give a (weak) lower bound for \(\pi(x)\).

1. Let \(p_n\) be the \(n\)-th prime. Show that \(p_{n+1} \leq p_1 \cdots p_n + 1\).

2. Deduce that for all \(n \geq 1\), we have \(p_n \leq 2^{2^n}\).

3. Conclude that for all \(x \geq 2\), one has \(\pi(x) \geq \log(\log x)\).