Midterm
October 23, 2019

The use of notes and electronic devices is forbidden. All answers have to be justified. You can use all results proved in lectures without proof. If you use a result seen in a homework or in recitation you need to prove it. You may use the back of the last sheet if you need extra space.

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Exercise 1. (12 points) Prove or disprove: for any of the following statements, say if it is true or false, by either proving it or providing a counterexample.

1. Let \(a, b, c\) be integers. If \(a\) divides \(bc\) then \(a\) divides \(b\) or \(c\).

2. Let \(G\) and \(H\) be commutative groups. Then \(G \times H\) is commutative.

3. Let \(G\) be a group and \(H\) a subgroup of \(G\). If \(G\) is commutative, then so is \(H\).

4. Let \(G\) be a group with identity element \(e\) and \(x \in G\) an element of order 4. Then \(x^{20} = e\).
Exercise 2. Let $G$ be the group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$.

1. (5 points) Write out the list of the elements of $G$, and for each element, give its order.

2. (2 points) What is the order of $G$?

3. (2 points) Is $G$ cyclic?
4. (2 points) Is $G$ isomorphic to $\mathbb{Z}/8\mathbb{Z}$?

5. (3 points) Give an example of a proper subgroup of $G$.

6. (4 points) We consider the map $p : G \to \mathbb{Z}/2\mathbb{Z}$ given by $p(x, y) = x$. Determine its kernel and image.
Exercise 3. Let $G$ be a group.

1. (2 points) What does it mean for $G$ to be commutative?

2. (2 points) What does it mean for $G$ to be cyclic?

3. (3 points) Show that if $G$ is cyclic, then $G$ is commutative.

4. (3 points) Give an example of a group which is commutative, but not cyclic.
Exercise 4. Let $*$ be the law of composition of $\mathbb{Z}$ given by $x * y = 2x + 2y$.

1. (4 points) Is it associative?

2. (2 points) Is it commutative?

3. (3 points) Does it have an identity element?
Exercise 5.  

1. (2 points) State Bézout’s theorem.

2. (5 points) Let $a, b, c$ be integers such that $\gcd(a,b) = 1$ and $\gcd(a,c) = 1$. Show that $\gcd(a, bc) = 1$. 

Exercise 6. Let $G$ be the set given by

$$G = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \in M_2(\mathbb{R}), \ a \in \{1, -1\}, b \in \mathbb{Z} \right\}.$$ 

1. (2 points) Show that $G$ is a subset of $GL_2(\mathbb{R})$, the set of invertible $2 \times 2$ matrices with real coefficients.

2. (8 points) Show that $G$ is a subgroup of $(GL_2(\mathbb{R}), \cdot)$. 
3. (3 points) Is $G$ commutative?

4. (5 points) Determine all elements of order 2 of $G$. 
Exercise 7. (6 points) If $a, b \in \mathbb{Z}$ are such that $a \equiv b \pmod{n}$, show that $\gcd(a, n) = \gcd(b, n)$. 