Algebra homework 1
Set theory, equivalence relations
Due September 18th, 2019

Please hand in your homework stapled, with your name written on it. All answers have to be justified.

Exercise 1. Let \( f : \mathbb{R} \rightarrow \mathbb{R} \) be the map \( f : x \mapsto (x + 1)^2 \). Compute the inverse image sets \( f^{-1}(A) \) of the following sets \( A \):

(a) \( \{ -9 \} \)

Solution. This is empty, as \(-9\) is never a square.

(b) \( \{ -1, 0, 4 \} \)

Solution. We have
\[
\begin{align*}
f^{-1}(\{ -1, 0, 4 \}) &= \{ x \in \mathbb{R} : (x + 1)^2 = -1, 0 \text{ or } 4 \} \\
&= \{ x \in \mathbb{R} : x + 1 = 0, 2 \text{ or } -2 \} \\
&= \{-1, 1, -3\}.
\end{align*}
\]

(c) \( [0, +\infty) = \{ x \in \mathbb{R} : x \geq 0 \} \)

Solution. \( f^{-1}([0, +\infty)) = \{ x \in \mathbb{R}, (x + 1)^2 \geq 0 \} = \mathbb{R} \).

Exercise 2. Let \( f : X \rightarrow Y \) be a map between sets.

1. For any two subsets \( A, B \) of \( Y \), show that
\[
\begin{align*}
f^{-1}(A) \cup f^{-1}(B) &= f^{-1}(A \cup B) \quad \text{and} \quad f^{-1}(A) \cap f^{-1}(B) &= f^{-1}(A \cap B).
\end{align*}
\]

Solution.
\[
\begin{align*}
f^{-1}(A) \cup f^{-1}(B) &= \{ x \in X \text{ such that } x \in f^{-1}(A) \text{ or } x \in f^{-1}(B) \} \\
&= \{ x \in X \text{ such that } f(x) \in A \text{ or } f(x) \in B \} \\
&= \{ x \in X \text{ such that } f(x) \in A \cup B \} \\
&= f^{-1}(A \cup B).
\end{align*}
\]

In exactly the same way,
\[
\begin{align*}
f^{-1}(A) \cap f^{-1}(B) &= \{ x \in X \text{ such that } x \in f^{-1}(A) \text{ and } x \in f^{-1}(B) \} \\
&= \{ x \in X \text{ such that } f(x) \in A \text{ and } f(x) \in B \} \\
&= \{ x \in X \text{ such that } f(x) \in A \cap B \} \\
&= f^{-1}(A \cap B).
\end{align*}
\]

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2. For any two subsets $A, B$ of $X$, show that

$$f(A) \cup f(B) = f(A \cup B).$$

**Solution.** Let $y \in Y$. We have $y \in f(A) \cup f(B)$ if and only if $y$ is of the form $f(x)$ where $x \in A$ or $x \in B$. This is the case if and only if $y = f(x)$ with $x \in A \cup B$, that is, if and only if $y \in f(A \cup B)$, whence the result.

3. (a) Show that in general

$$f(A) \cap f(B) \neq f(A \cap B)$$

by giving a counterexample. (Hint: draw a picture)

**Solution.** The following picture shows that if $X = \{a, b\}$ is a set with two elements, $Y = \{y\}$ is a singleton (that is, a set with one element), and $f$ is defined to be the constant map, we have, putting $A = \{a\}$ and $B = \{b\}$, that

$$f(A) \cap f(B) = \{y\}$$

whereas

$$f(A \cap B) = f(\emptyset) = \emptyset.$$  

Note that the inclusion

$$f(A) \cap f(B) \supseteq f(A \cap B)$$

is nevertheless always true. Indeed, if $y \in f(A \cap B)$, then we can write $y = f(x)$ with $x \in A \cap B$ (that is, $x \in A$ and $x \in B$), which means in particular that $y \in f(A)$ and $y \in f(B)$, that is, $y \in f(A) \cap f(B)$.

(b) Show that we do get equality in (1) if we furthermore assume that $f$ is injective.
Solution. The answer to the previous question illustrates the fact that non-injectivity makes the equality go wrong. Assume that \( f \) is injective. We already know that
\[
f(A) \cap f(B) \supseteq f(A \cap B)
\]
so it suffices to prove that
\[
f(A) \cap f(B) \subseteq f(A \cap B).
\]
If \( y \in f(A) \cap f(B) \) then \( y \in f(A) \) and \( y \in f(B) \), that is, \( y = f(a) = f(b) \) for some \( a \in A \) and some \( b \in B \). Since \( f \) is injective, we have \( a = b \). Thus, \( a \in B \), and so \( a \in A \cap B \), which implies that \( y \in f(A \cap B) \).

Exercise 3. Let \( f : X \to Y \) and \( g : Y \to Z \) be maps between sets.

1. Show that if \( g \circ f \) is injective, then \( f \) is injective.

   **Solution.** Assume that \( g \circ f \) is injective. Let \( x, y \in X \) be such that \( f(x) = f(y) \). Apply \( g \) to both sides of the equation, to get \( g(f(x)) = g(f(y)) \). By injectivity of \( g \circ f \), we then get \( x = y \). This proves \( f \) is injective.

2. Show that if \( g \circ f \) is surjective, then \( g \) is surjective.

   **Solution.** Assume \( g \circ f \) is surjective. Let \( z \in Z \). By surjectivity of \( g \circ f \), we have an element \( x \in X \) such that \( g(f(x)) = z \). Then \( y = f(x) \) gives us an element of \( Y \) such that \( g(y) = z \), so \( g \) is surjective.

Exercise 4. For an element \( x = (x_1, x_2) \) of the plane \( \mathbb{R}^2 \), we denote by \( \|x\| = \sqrt{x_1^2 + x_2^2} \) its Euclidean norm. Let \( \sim \) be the relation on the plane \( \mathbb{R}^2 \) given by
\[
x \sim y \quad \text{if} \quad \|x\| = \|y\|.
\]
Show that \( \sim \) is an equivalence relation and describe its equivalence classes.

**Solution.** For any \( x \in \mathbb{R}^2 \), we have \( \|x\| = \|x\| \), so \( x \sim x \), so \( \sim \) is reflexive. For any \( x, y \in \mathbb{R}^2 \), if we have \( \|x\| = \|y\| \) then we have \( \|y\| = \|x\| \), and therefore \( x \sim y \) implies \( y \sim x \), which means that \( \sim \) is symmetric. Finally, for any \( x, y, z \in \mathbb{R}^2 \), if \( x \sim y \) and \( y \sim z \), then we have \( \|x\| = \|y\| \) and \( \|y\| = \|z\| \), and therefore \( \|x\| = \|z\| \), that is, \( x \sim z \), so that \( \sim \) is transitive.

Let \( x \in \mathbb{R}^2 \), and put \( r = \|x\| \). Then the equivalence class of \( x \) is the set
\[
\{ y \in \mathbb{R}^2; \|y\| = r \}
\]
of all elements with norm \( r \). If \( r > 0 \) this is the circle \( C_r \) of radius \( r \) centered in the origin. For \( r = 0 \), the only point of norm zero is the origin, so the corresponding equivalence class is just the singleton \( \{(0, 0)\} \).

Exercise 5. We define a relation \( R \) on \( \mathbb{Z} \) by \( aRb \) if \( a \) divides \( 2b \).
1. Is $R$ reflexive?

   *Solution.* Yes, since for every $a \in \mathbb{Z}$, we do have $a|2a$.

2. Is it symmetric?

   *Solution.* No: we have $2R8$, but we do not have $8R2$.

3. Is it transitive?

   *Solution.* If the relation were transitive, we would have that if $a$ divides $2b$ and $b$ divides $2c$, then $a$ divides $2c$. Intuitively, this seems wrong as the assumption should a priori just imply that $a$ divides $4c$, not $2c$. Let us build a counterexample based on this intuition, by trying $c$ to be as small as possible while $a$ is divisible by 4, so that we really need the factor 4. Thus, $a = 4$, $b = 2$, $c = 1$ gives us a counterexample.