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Stochastic approximations to curve-shortening flows via particle systems

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Abstract

Curvature-driven flows have been extensively considered from a deterministic point of view. Besides their mathematical interest, they have been shown to be useful for a number of applications including crystal growth, flame propagation, and computer vision. In this paper, we describe a random particle system, evolving on the discretized unit circle, whose profile converges toward the Gauss–Minkowsky transformation of solutions of curve-shortening flows initiated by convex curves. Our approach may be considered as a type of stochastic crystalline algorithm. Our proofs are based on certain techniques from the theory of hydrodynamical limits.

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1. Introduction and statement of results

1.1. Curvature-driven flows

Let $\mathscr{C}(p,t) : S^1 \times [0,T) \mapsto \mathbb{R}^2$ be a family of embedded curves where t parameterizes the family and p parameterizes each curve. In this paper, we will consider stochastic interpretations of certain *curvature-driven flows*, i.e. starting from an initial embedded curve $\mathscr{C}_0(p)$ we consider the solution (when it exists) of an equation of the form

$$\frac{\partial \mathscr{C}(p,t)}{\partial t} = \hat{V}(\kappa(p,t))\mathscr{N}, \quad \mathscr{C}(\cdot,0) = \mathscr{C}_0(\cdot), \tag{1.1}$$

where $\kappa(p,t)$ denotes the curvature and $\mathscr{N}$ denotes the inner unit normal of the curve $\mathscr{C}(\cdot,t)$ at p. Of particular interest is the case in which $\hat{V}(x) = \pm x^\alpha$.

The case $\hat{V}(x) = x$ corresponds to the *Euclidean* curve-shortening flow [7] while $\hat{V}(x) = x^{1/3}$ corresponds to the *affine* curve shortening, which is of strong relevance in computer vision and image processing [14]. The literature on these flows is extensive, for a recent review see [5].

We should note that these latter flows are particularly important since they are *gradient flows*. Indeed, for $\alpha = 1$ the equation may be shown to be direction in which curve length is shrinking as fast as possible using only local information. The equation is also a *geometric heat equation* since it may be written in terms of the Euclidean arc length ds as

$$\frac{\partial \mathscr{C}}{\partial t} = \frac{\partial^2 \mathscr{C}}{\partial s^2}.$$

Similar remarks apply to the case $\alpha = \frac{1}{3}$ since here area is shrinking as fast as possible with respect to affine arc length, and one may formulate the flow as an *affine invariant heat equation* by taking the two derivatives with respect to the affine invariant arc length [14]. Since in both cases, we get gradient flows and resulting heat equations, a stochastic interpretation seems quite natural.

Since we will be dealing with convex curves in this paper, we employ the standard parameterization via the Gauss map, that is fixing $p = \theta$, the angle between the exterior normal to the curve and a fixed axis. It is well known that the Gauss map can be used to map smooth convex curves $\mathscr{C}(\cdot)$ into positive functions $m(\cdot)$ on S^1 such that $\int_{S^1} e^{2\pi i\theta} m(\theta)\, d\theta = 0$, and that this map can be extended to the *Gauss–Minkowsky* bijection between convex curves with $\mathscr{C}(0) = 0$ and positive measures on S^1 with zero barycenter; see [4, Section 8] for details. We denote by $\mathscr{M}_+^0$ the latter set of measures.

Under this parameterization, a convex curve $\mathscr{C}(\theta)$ can be reconstructed from a $\mu \in \mathscr{M}_+^0$ by the formula

$$\mathscr{C}(\theta) = \int_0^\theta e^{2\pi i\Theta} \mu(d\Theta), \tag{1.2}$$

using linear interpolation over jumps of the function $\mathscr{C}(\theta)$. Further, whenever μ possesses a strictly positive density $m(\theta)\,d\theta$ then the curvature of the curve at θ is $\kappa(\theta) = 1/m(\theta)$.

Another useful property in working with measures $\mu \in \mathscr{M}_+^0$ is that the evolution of the density $m(\cdot)$ takes a particularly simple form: indeed, one gets (see e.g. [16, Eqs. (1.1), (1.2)])

$$\frac{\partial m(t,\theta)}{\partial t} = -\frac{\partial^2 V(m(t,\theta))}{\partial \theta^2} - V(m(t,\theta)), \quad V(x) := \hat{V}(1/x). \tag{1.3}$$

There are a number of interesting special cases. For example, when $\hat{V}(x) = -x^{-1}$ gives the linear evolution

$$m_t = m_{\theta\theta} + m.$$

In this case, we may separate variables as in the usual analysis of the heat equation and see that as $t \to \infty$, $m(\theta, t)$ goes to constant, and thus the initial curve asymptotically approaches a circle (of infinite radius) [12]. Hence, for this curvature-driven flow there is no blowup. (See also [1] for various results about expanding flows.) For $\hat{V} = 1$, the equation becomes

$$m_t = -1$$

which has solution

$$m(t,\theta) = -t + m(0,\theta).$$

Thus, here we get blowup in finite time (for the curve) when $t = m(0,\theta)$.

In general, for $\hat{V}(x) = x^\alpha$, $\alpha \geqslant 0$, Eq. (1.3) becomes

$$\frac{\partial m(t,\theta)}{\partial t} = -\frac{\partial^2 m^{-\alpha}(t,\theta)}{\partial \theta^2} - m^{-\alpha}(t,\theta) \tag{1.4}$$

which is defined up to a finite time, at which singularities may develop. For $\alpha = 1$, at the blowup time the curve has shrunk to a "circular point" (see [7]), for $\alpha = \frac{1}{3}$ it has shrunk to an "ellipsoidal-shaped" point (see [14]), whereas for $\alpha < \frac{1}{3}$ singularities may develop earlier. Indeed, in this regime, the aspect ratio of the evolving curve goes to infinity as the curve shrinks [2, Theorem 2] for a generic initial curve. The regime $\alpha \in (\frac{1}{3}, 1)$ has been considered in [1,15], with results similar to those of $\alpha = 1$. Since for $\alpha \geqslant 0$, the length of the evolving curve decreases, we will refer to flows with speed functions of the form $\hat{V}(x) = x^\alpha$, $\alpha \geqslant 0$ as *curve-shortening flows.*

1.2. Stochastic approximations

Our interest is in constructing stochastic approximations to the solutions of Eqs. (1.4). Approximations corresponding to polygonal curves have been discussed in the literature under the name "crystalline motion", see [16] for a description of recent results and references. Our approach is different and can be thought of as a *stochastic* crystalline algorithm: we will construct a stochastic particle system whose profile defines an *atomic* measure on S^1, such that the corresponding curve is a convex polygon. Applying tools from hydrodynamic limits, we then prove that the (random) evolution of this polygonal curve converges, in the limit of a large number of particles, to curve evolution under the curve-shortening flow. This approach is related in spirit but not in techniques to recent work on particle systems which approximate the non-linear filtering equations; see [6] and references therein.

Our work is motivated by the fact (see [17]) that the uniform measure on the (finite) set of convex polygons of area bounded by 1 which encircle the origin and possesses vertices on the lattice $n^{-1}\mathbb{Z}^2$ satisfies a large deviation principle with rate function related to the affine length of curves. This suggests that natural (random) dynamics for these polygons should be related to evolution according to affine curve shortening, i.e. to solutions of (1.4) with $\alpha = \frac{1}{3}$. The system we construct here is a first step in the study of this relationship.

We conclude this introduction by describing a particular case of our general result Theorem 3: fix $\varepsilon > 0$, consider the discrete torus T_N and, at time 0, put at each site i, $\eta_0(i)$ particles. Evolve the configuration $\eta_t(\cdot)$ in time such that each particle at site i jumps to one of its neighbors at rate $\varepsilon^{-2}N^2$ if $\eta_t(i) = 1$ and $\varepsilon^{-1}N^2/\eta_t(i)$ otherwise, dies at rate ε^{-2} if $\eta_t(i) = 1$, and gives birth at rate $\varepsilon^{-2}/2$ if $\eta_t(i) = 2$. Define the (random) measure $\mu_t^{\varepsilon,N} = N^{-1}\sum_{i\in T_N}\eta_t(i)\delta_{i/N}$ on S^1, add (at most two) atoms at $0, \pi, \pm\pi/2$ to create a $\bar{\mu}_t^{\varepsilon,N}$ with zero barycenter, and construct from that measure a curve $\mathscr{C}_{N,\varepsilon}(t,\cdot)$ as explained in (1.2). Then, if $\mathscr{C}_{N,\varepsilon}(0,\cdot)$ converges as $N\to\infty$ to a smooth strictly convex curve $\mathscr{C}_0(\cdot)$, then as first $N\to\infty$ and then $\varepsilon\to 0$ it holds that $\mathscr{C}_{N,\varepsilon}(t,\cdot)$ converges (in Hausdorff distance, say) to the solution of the Euclidean curve shortening (1.1) with $\alpha = 1$.

The structure of this paper is as follows: Section 2 presents some approximation results for quasilinear parabolic equations and their relation to curve shortening. Section 3 introduces our particle system, states the general hydrodynamic limit result Theorem 2 which is at the heart of our approach, states the main curve convergence result Theorem 3, and provides a family of stochastic evolutions which satisfy our assumptions and correspond to curve-shortening equations with $1/\alpha$ integer. Finally, Section 4 presents the proofs of our claims.

2. PDE approximations

We present in this section, a general result concerning the existence and uniqueness of a certain class of quasilinear parabolic equations, and show how such equations are approximations of the curve-shortening equations described above. Let $\Phi, V : \mathbb{R}_+ \mapsto \mathbb{R}$ satisfy the following:

Assumption A. (A.1) $\Phi \in C^3(\mathbb{R}_+)$, $V \in C^1(\mathbb{R}_+)$.

(A.2) For every $L > 0$ there exist constants $c_L, d_L > 0$ such that

$$\min_{x \in [0,L]} \Phi'(x) \geqslant c_L, \quad \max_{x \in [0,L]} |\Phi''(x)| \leqslant d_L.$$

(A.3) $V(\cdot)$ is bounded and $V(0) \geqslant 0$.

Define the operator $L : C^{1,2}(\mathbb{R}_+ \times S^1) \mapsto C(\mathbb{R}_+ \times S^1)$ as

$$L\rho(t,x) = -\partial_t \rho(t,x) + \tfrac{1}{2}\partial_{xx}\Phi(\rho)(t,x) + V(\rho(t,x)). \tag{2.1}$$

The basic existence and uniqueness result alluded to above is the following (classical) proposition, whose proof is given for completeness in the appendix.

Proposition 1. *Suppose* Φ, V *satisfy Assumption* A, *and let* $m(\cdot) \in C^{2+\beta}(S^1)$, *for some* $1 \geqslant \beta > 0$, *be a strictly positive function. Then there exists a unique solution* $\rho \in C^{2+\beta}(S^1)$ *to the equation*

$$L\rho(t,x) = 0, \quad \rho(0,x) = m(x). \tag{2.2}$$

Further, $\rho(t,x)$ *is strictly positive.*

Note that the curve-shortening flow (1.4) is not covered by Proposition 1, for the functions $V(x) = \Phi(x) = -x^{-\alpha}$ do not satisfy Assumption A (and indeed, the curve-shortening flow does possess a finite blowup time, contrary to the conclusion of Proposition 1). We thus wish to approximate this flow, e.g. by using functions of the form $\Phi_{\alpha,\varepsilon}(x) = 1/\varepsilon - 1/(x + \varepsilon^{1/\alpha})^{\alpha}$ and $V_{\alpha,\varepsilon}(x) = -x/(x + \varepsilon^{1/\alpha})^{\alpha+1}$ (see Section 3.3). We thus establish next a convergence result for solutions of quasilinear parabolic equations that approximate curve-shortening equations. In what follows, set $\mathbb{R}_+^0 = (0, \infty)$.

Theorem 1. *Suppose functions* $\Phi \in C^2(\mathbb{R}_+^0)$, $V \in C^1(\mathbb{R}_+^0)$ *and* $m \in C^{2+\beta}(S^1)$ *are given such that* $m(\cdot)$ *is strictly positive and* (2.2) *holds on* $[0,T)$ *with* ρ *strictly positive. Let* $\Phi_\varepsilon, V_\varepsilon$ *satisfy Assumption* A *and assume that* $\Phi'_\varepsilon, \Phi''_\varepsilon, V_\varepsilon$ *converge uniformly on compact subsets of* $(0,\infty)$ *to* Φ', Φ'', V. *Let* L^ε *denote the operator* L *with the functions* $\Phi_\varepsilon, V_\varepsilon$ *substituted for the functions* Φ, V, *and let* $\rho_\varepsilon(t,x)$ *satisfy* $L^\varepsilon \rho_\varepsilon(t,x) = 0$, $\rho_\varepsilon(0,x) = m(x)$. *Then, for any* $\delta > 0$,

$$\limsup_{\varepsilon \to 0} \sup_{(t,x) \in [0,T-\delta] \times S^1} \frac{\rho_\varepsilon(t,x)}{\rho(t,x)} = \limsup_{\varepsilon \to 0} \sup_{(t,x) \in [0,T-\delta] \times S^1} \frac{\rho(t,x)}{\rho_\varepsilon(t,x)} = 1. \tag{2.3}$$

For the proof, we refer to the appendix. Note that in Theorem 1, we did *not* assume that Φ, V satisfy Assumption A. On the other hand, the existence and uniqueness of $\rho^\varepsilon(t,x)$ is assured by Proposition 1.

3. Particle systems, hydrodynamical limits, and approximate curvature flows

We construct in this section the particle systems alluded to above, prove their hydrodynamical limits, and relate them to approximate curvature flows.

3.1. Birth and death zero range particle systems and hydrodynamic limits

Let $T_N = \mathbb{Z} \backslash N\mathbb{Z}$ denote the discrete torus. Let $g : \mathbb{N} \to \mathbb{R}_+$ (the *jump rate*, with $g(0) = 0$), $b : \mathbb{N} \to \mathbb{R}_+$ (the *birth rate*), $d : \mathbb{N} \to \mathbb{R}_+$ (the *death rate*, with $d(0) = 0$) be given, and define the Markov generator on the particle configuration $E_N = \mathbb{N}^{T_N}$ by

$$(\mathscr{L}^N f)(\eta) = N^2(\mathscr{L}_0 f)(\eta) + (\mathscr{L}_1 f)(\eta), \quad f \in C_b(E_N),$$

where

$$(\mathscr{L}_0 f)(\eta) = \frac{1}{2} \sum_{i \in T_N} g(\eta(i))[f(\eta^{i,i+1}) + f(\eta^{i,i-1}) - 2f(\eta)],$$
$$(\mathscr{L}_1 f)(\eta) = \sum_{i \in T_N} [b(\eta(i))[f(\eta^{i,+}) - f(\eta)] + d(\eta(i))[f(\eta^{i,-}) - f(\eta)]],$$

and

$$\eta^{i,i\pm 1}(j) = \begin{cases} \eta(j) + 1, & j = i \pm 1, \eta(i) \neq 0, \\ \eta(j) - 1, & j = i, \eta(i) \neq 0, \\ \eta(j), & \text{else} \end{cases},$$

$$\eta^{i,+}(j) = \begin{cases} \eta(j) + 1, & j = i, \\ \eta(j), & \text{else} \end{cases} \quad \eta^{i,-}(j) = \begin{cases} \eta(j) - 1, & j = i, \eta(i) > 0, \\ \eta(j), & \text{else.} \end{cases}$$

In words, under $\mathscr{L}^N$, each particle at location i jumps to one of its neighboring locations at rate $N^2 g(\eta(i))/\eta(i)$, dies at rate $d(\eta(i))/\eta(i)$, and a new particle is created at location i with rate $b(\eta(i))$. Thus, we deal here with zero range processes in the presence of births and deaths.

We use S_t^N to denote the associated Markov semigroup, and we denote by $\mu_{t,N}$ the law of the process at time t, with initial law $\mu_{0,N}$, under this Markovian semigroup. We also use μ^N to denote the law of the *trajectory* of the process.

In order to state our main limit result, we need to introduce the appropriate equilibrium measure, as in [9, Chapter 2.3]. Define $Z : \mathbb{R}_+ \to \mathbb{R}_+ \cup \{+\infty\}$ by $Z(\varphi) := \sum_k \frac{\varphi^k}{g(k)!}$, where $g(k)! = g(1) \cdots g(k)$ and $g(0)! = 1$. Set $\mathscr{D}_g = \{\varphi \in \mathbb{R}_+ : Z(\varphi) < \infty\}$, and $\varphi^* = \sup\{\varphi : \varphi \in \mathscr{D}_g\}$. For any $\varphi \in \mathscr{D}_g$, we define the probability measure $\bar{p}_\varphi$

on $\mathbb{N}$ by

$$\bar{p}_{\varphi}(k) = \frac{\varphi^k}{g(k)!Z(\varphi)},$$

and set $R(\varphi) := \varphi \dfrac{Z'(\varphi)}{Z(\varphi)}, \varphi \in \mathscr{D}_g$ (see [9, pp. 28–31] for background).

Throughout this section, we always make the following hypotheses on $g(\cdot)$.

Assumption B. (B.1) $\inf_{k \geqslant 1} g(k) > 0$, and $\limsup_{k \to \infty} \frac{g(k)}{k} = 0$.

(B.2) $Z(\varphi) \nearrow_{\varphi \nearrow \varphi^*} \infty$.

(B.3) There exists a constant $C_1 < \infty$ such that $\limsup_{k \to \infty} [g(k)b(k-1) - b(k) + \frac{d(k+1)}{g(k+1)} - d(k)] \leqslant C_1$ and $\sup_k |b(k)| \leqslant C_1, \sup_k |d(k)| \leqslant C_1$.

The following basic properties of $\bar{p}_{\varphi}$, proved in [9, pp. 28–31], are crucial in the sequel.

Lemma 1. *Let Assumption* (B.1) *hold. Then,*

(a) $\varphi^* > 0, R(\varphi) \nearrow_{\varphi \nearrow \varphi^*} \infty$, *and for each* $\varphi < \varphi^*$ *there exists a* $\theta(\varphi) > 0$ *such that* $\bar{p}_{\varphi}$ *possesses exponential moments with parameter* $\theta(\varphi)$.

(b) *Set* $\Phi(\alpha) = R^{-1}(\alpha)$ *and* $p_\alpha = \bar{p}_{\Phi(\alpha)}$. *Then,* $\Phi(\cdot)$ *is a smooth function with strictly increasing derivative,* $\Phi'(0) \in (0, \infty)$, *and*

$$E_{p_\alpha}(X) = \alpha, \quad E_{p_\alpha}(g(X)) = \Phi(\alpha).$$

(c) *Set* $\nu_\alpha = p_\alpha^{\otimes \mathbb{Z}}$ *and let* $\nu_{\alpha,N}$ *denote the restriction of* ν_α *to* T_N. *Then* $\nu_{\alpha,N}$ *is reversible, and hence invariant, for the Markov generator* L_0^N.

In the sequel, for any function h defined on $\mathbb{N}$, we set $\tilde{h}(\alpha) := E_{p_\alpha}(h(X))$. In particular, by Lemma 1, $\tilde{g}(\alpha) = \Phi(\alpha)$. We need below the following assumption on the initial law of our Markov evolution:

Assumption C. There exists a $\delta > 0$ and an $m \in C^{2+\delta}(S^1)$ strictly positive such that

$$\frac{1}{N} H\left(\mu_{0,N} \middle| \prod_{i=0}^{N-1} p_{m(\frac{i}{N})}\right) \xrightarrow[N \to \infty]{} 0.$$

Set

$$V(\alpha) = V_+(\alpha) - V_-(\alpha) := \tilde{b}(\alpha) - \tilde{d}(\alpha).$$

Let $\rho(t,x):[0,T]\times S^1\mapsto\mathbb{R}_+$ denote a $C^{1,2+\delta}$ strictly positive solution of the PDE

$$\partial_t\rho(t,x)=\tfrac{1}{2}\partial_{xx}\Phi(\rho)(t,x)+V(\rho)(t,x),\quad \rho(0,x)=m(x). \tag{3.1}$$

(When Assumption B is in force, such a solution exists and is unique by Proposition 1 above since $\infty>\Phi'(\cdot)>0$ and $V(\cdot)$ is a smooth bounded function). We are now ready to state the hydrodynamic limit result for the laws $\mu_{t,N}$:

Theorem 2. *Let Assumptions* B *and* C *hold. Then, for any function* $G\in C(S^1)$, *any* $\delta>0$, *and any* $t\in[0,T]$,

$$\lim_{N\to\infty}\ \mu_{t,N}\left\{\eta:\ \left|\frac{1}{N}\sum_{i\in T_N}\eta(i)G\left(\frac{i}{N}\right)-\int_{S^1}G(x)\rho(t,x)\,dx\right|>\delta\right\}=0.$$

Remark. We note that in the terminology of [9], g satisfies a SLG assumption but does not satisfy the FEM assumption and is not attractive. This requires some additional work in deriving the hydrodynamic limits.

3.2. Stochastic curve-shortening convergence

We begin by explicitly constructing random polygons from particle configurations. Each particle configuration $\eta(\cdot)$ defines a positive measure on S^1 by $\mu_\eta=\sum_{k\in T_N}\eta(k)\delta_{2\pi k/N}$. Unfortunately, this measure does not possess necessarily a zero barycenter, and thus does not correspond a priori to a closed convex curve. To remedy this situation, set

$$b_\eta=b_\eta^{\mathrm{R}}+ib_\eta^{\mathrm{I}}=\sum_{k\in T_N}e^{2\pi k/N}\eta(k),$$

and define

$$\bar{\mu}_\eta=\mu_\eta+|b_\eta^{\mathrm{R}}|\delta_{\pi/2+(\pi/2)\mathrm{sign}(b_\eta^{\mathrm{R}})}+|b_\eta^{\mathrm{I}}|\delta_{-(\pi/2)\mathrm{sign}(b_\eta^{\mathrm{I}})}.$$

Then $\bar{\mu}_\eta\in\mathscr{M}_+^0$, and it defines a curve by a linear interpolation between the jump points of the function $C_\eta(\theta)=\int_0^\theta e^{2\pi i\Theta}\bar{\mu}_\eta(d\Theta)$.

Fix next $\alpha>0$, consider the functions $\Phi_\alpha(x)=-x^{-\alpha}$, $V_\alpha(x)=-x^{-\alpha}$, and define the operator L_α as in (2.1). Fix an m satisfying Assumption C, and let ρ_α denote the solution of (2.2) with operator L_α, with blowup time T_α, and associated curve $\mathscr{C}_\alpha(t,\theta)$. Let $g_{\alpha,\varepsilon},b_{\alpha,\varepsilon},d_{\alpha,\varepsilon}$ satisfy Assumption B, set $\Phi_{\alpha,\varepsilon}$ and $V_{\alpha,\varepsilon}$ as in Section 3.1. The following assumption is needed in order to relate the particle system with the curve-shortening flow:

Assumption D. (D.1) $\Phi_{\alpha,\varepsilon}$, $V_{\alpha,\varepsilon}$ satisfy Assumption A.

(D.2) $\Phi'_{\alpha,\varepsilon}, \Phi''_{\alpha,\varepsilon}, V_{\alpha,\varepsilon}$ converge uniformly on compact subsets of $(0,\infty)$ to $\Phi'_{\alpha}, \Phi''_{\alpha}, V_{\alpha}$.

Our main result is the following:

Theorem 3. *Let $\mathscr{C}^N_{\alpha,\varepsilon} : \mathbb{R}_+ \times S^1 \mapsto \mathbb{R}_+$ denote the curve corresponding to the particle system defined above. Fix $\delta, \delta' > 0$. Then,*

$$\lim_{\varepsilon \to 0} \limsup_{N \to \infty} P\left(\sup_{(t,\theta) \in [0, T_\alpha - \delta] \times S^1} |\mathscr{C}^N_{\alpha,\varepsilon}(t,\theta) - \mathscr{C}_\alpha(t,\theta)| > \delta' \right) = 0. \tag{3.2}$$

If further $\mathscr{C}_\alpha(t,\theta) \to_{t \to T_\alpha} 0$, $\mathscr{C}_\alpha(t,\theta) \coloneqq 0$ for $t > T_\alpha$, and there exists a $z_0 = z_0(\alpha)$ such that $\Phi'_{\alpha,\varepsilon}(z) \geqslant 0$, $V_{\alpha,\varepsilon}(z) < 0$ for all $0 < z < z_0$, then $T_\alpha - \delta$ in (3.2) *can be replaced by any deterministic constant $T > 0$.*

Proof. Eq. (3.2) is a straightforward consequence of Theorems 1 and 2, the fact that the function $e^{2\pi i\theta}$ is continuous, and the regularity of $\mathscr{C}_\alpha(t,\cdot)$. To see the second part of the claim, let $\rho_{\varepsilon,\alpha}(t,x)$ denote the solution of (3.1) with the functions $\Phi_{\alpha,\varepsilon}$ and $V_{\alpha,\varepsilon}$, and set $\mu_{\alpha,\varepsilon}(t) \coloneqq \max_{x \in S^1} \rho_{\alpha,\varepsilon}(t,x)$. We claim first that there exists a δ_1 and an ε_0 such that for all $\varepsilon < \varepsilon_0$,

$$\mu_{\alpha,\varepsilon}(t_0) < \delta_1, \quad \text{some } t_0 \Rightarrow \mu_{\alpha,\varepsilon}(t) < \mu_{\alpha,\varepsilon}(t_0) < \delta_1, \ \ \forall t > t_0. \tag{3.3}$$

This implies the second part of the claim since by Theorem 2,

$$\limsup_{N \to \infty} P\left(\sup_{(t,\theta) \in [0,T] \times S^1} |\mathscr{C}^N_{\alpha,\varepsilon}(t,\theta) - \mathscr{C}_{\alpha,\varepsilon}(t,\theta)| > \delta' \right) = 0$$

while $\lim_{\varepsilon \to 0} \lim_{t \to T_\alpha} \mu_{\alpha,\varepsilon}(t) = 0$.

To see (3.3), note that by the assumptions, one may find a ε_0 and a δ_1 such that

$$\forall \varepsilon < \varepsilon_0, \quad 0 < z < \delta_1: \ V_{\alpha,\varepsilon}(z) < 0, \Phi'_{\alpha,\varepsilon}(z) \geqslant 0.$$

Suppose (3.3) does not hold. Then there exists a $t_1 \in (t_0, t), s_1 \in S^1$ with $\partial_t \rho_{\alpha,\varepsilon}(t_1, s_1) = 0$, $\partial_x \rho_{\alpha,\varepsilon}(t_1, s_1) = 0$, $\partial_{xx} \rho_{\alpha,\varepsilon}(t_1, s_1) \leqslant 0$ while $V(\rho_{\alpha,\varepsilon}(t_1, s_1)) < 0$, contradicting (3.1). $\square$

Remark. Note that for Theorem 3 we have that $\mathscr{C}_\alpha(t,\theta) \to_{t \to T_\alpha} 0$ when $\alpha \in [\frac{1}{3}, 1]$.

3.3. Approximate curvature flows

We now present different candidates for the functions b, d, g defining the particle systems of Section 3.1. The first two relate to an approximate version of the Euclidean curvature flow, while the last one relates to a general curve-shortening flow of parameter α with $1/\alpha$ integer. Throughout, $\varepsilon > 0$ is a fixed parameter, and we set $W(\varphi) = V(R(\varphi))$.

I. *Approximate Euclidean curvature flow.* Set

$$\Phi_{\varepsilon,1}(r) = \frac{1}{\varepsilon} - \frac{1}{r+\varepsilon}.$$

Then, $R_{\varepsilon,1}(\varphi) = \varepsilon(1/(1-\varepsilon\varphi) - 1)$, and $Z_{\varepsilon,1}(\varphi) = (1-\varepsilon\varphi)^{-\varepsilon}$. Expanding, one finds that

$$g_{\varepsilon,1}(1) = \varepsilon^{-2}, \quad g_{\varepsilon,1}(k) = \frac{k}{\varepsilon(k-1+\varepsilon)}, \quad k \geqslant 2. \tag{3.4}$$

Choosing now $V_{\varepsilon,1}(r) = -r/(r+\varepsilon)^2$, one may compute the functions b, d by noting that with $W_{\varepsilon,1}(\varphi) = V_{\varepsilon,1}(R_{\varepsilon,1}(\varphi)) = -\varphi(1-\varepsilon\varphi)$, it must hold that

$$W_{\varepsilon,1}(\varphi) = -\varphi + \varepsilon\varphi^2 = \frac{1}{Z_{\varepsilon,1}(\varphi)} \sum_{k=0}^{\infty} (b_{\varepsilon,1}(k) - d_{\varepsilon,1}(k)) \frac{\varphi^k}{g_{\varepsilon,1}(k)!}.$$

Expanding, one finds that a possible choice for the birth and death rates is

$$b_{\varepsilon,1}(0) = b_{\varepsilon,1}(1) = 0, \quad b_{\varepsilon,1}(k) = \frac{(1-\varepsilon)k}{\varepsilon(\varepsilon+k-1)(\varepsilon+k-2)}, \quad k \geqslant 2 \tag{3.5}$$

and

$$d_{\varepsilon,1}(0) = 0, \quad d_{\varepsilon,1}(1) = \varepsilon^{-2}, \quad d_{\varepsilon,1}(k) = 0, \quad k \geqslant 2. \tag{3.6}$$

Note that for fixed $\varepsilon > 0$, the coefficients $g_{\varepsilon,1}(\cdot), 1/g_{\varepsilon,1}(\cdot), b_{\varepsilon,1}(\cdot), d_{\varepsilon,1}(\cdot)$ are uniformly bounded, and hence satisfy Assumption B.

II. *A simpler approximate Euclidean curvature flow.* The jump rate, birth and death coefficients described above suggest a further approximation of the Euclidean curvature flow: Set

$$\begin{aligned} &\bar{g}_{\varepsilon,1}(1) = \varepsilon^{-2}, \quad \bar{g}_{\varepsilon,1}(k) = \varepsilon^{-1}k/(k-1), \quad k \geqslant 2, \\ &\bar{b}_{\varepsilon,1}(2) = 2\varepsilon^{-2}, \quad \bar{b}_{\varepsilon,1}(k) = 0, \quad k \neq 2, \\ &\bar{d}_{\varepsilon,1}(1) = \varepsilon^{-2}, \quad \bar{d}_{\varepsilon,1}(k) = 0, \quad k \neq 1. \end{aligned} \tag{3.7}$$

Note that the coefficients in (3.7) are globally bounded, and hence satisfy Assumption B. Further, one finds that $\bar{Z}_{\varepsilon,1}(\varphi) = 1 - \varepsilon \log(1 - \varepsilon\varphi)$, and thus that

$$\bar{R}_{\varepsilon,1}(\varphi) = \frac{\varepsilon^2 \varphi}{(1-\varepsilon\varphi)(1-\varepsilon\log(1-\varepsilon\varphi))}.$$

Defining $\bar{\Phi}_{\varepsilon,1}(r) = \bar{R}^{-1}_{\varepsilon,1}(r)$, one sees that again, for ε small, $\bar{\Phi}_{\varepsilon,1}(r) \sim \varepsilon^{-1} - 1/r$, in the sense that for each $r_0 > 0$,

$$\limsup_{\varepsilon\to 0} \sup_{r>r_0} \left| \bar{\Phi}_{\varepsilon,1}(r) - \frac{1}{\varepsilon} + \frac{1}{r} \right| = 0.$$

One further notes that

$$\bar{\Phi}'_{\varepsilon,1}(r) = \frac{1}{r^2} \frac{\varepsilon^2 \bar{\Phi}^2_{\varepsilon,1}(r)}{1 - \varepsilon^2 \bar{\Phi}_{\varepsilon,r}(r) - \varepsilon \log(1 - \varepsilon \bar{\Phi}_{\varepsilon,1}(r))},$$

concluding that

$$\limsup_{\varepsilon\to 0} \sup_{r>r_0} \left| \bar{\Phi}'_{\varepsilon,1}(r) - \frac{1}{r^2} \right| = 0, \quad \limsup_{\varepsilon\to 0} \sup_{r>r_0} \left| \bar{\Phi}''_{\varepsilon,1}(r) + \frac{2}{r^3} \right| = 0.$$

Further, recalling the definition $\bar{W}_{\varepsilon,1}(\varphi) = \bar{V}_{\varepsilon,1}(\bar{R}_{\varepsilon,1}(\varphi))$, one finds that

$$\bar{W}_{\varepsilon,1}(\varphi) = -\bar{R}_{\varepsilon,1}(\varphi)(\varepsilon^{-1} - \varphi)^2,$$

and hence, $\bar{V}_{\varepsilon,1}(r) = -r(1 - \varepsilon\bar{\Phi}_{\varepsilon,1}(r))^2/\varepsilon^2$, implying by the above that

$$\limsup_{\varepsilon\to 0} \sup_{r>r_0} \left| \bar{V}_{\varepsilon,1}(r) + \frac{1}{r} \right| = 0.$$

III. *An approximate curve-shortening flow.* Fix $L \coloneqq 1/\alpha$ an integer, and set

$$\Phi_{\varepsilon,\alpha}(r) = \frac{1}{\varepsilon} - \frac{1}{(r+\varepsilon^L)^\alpha}.$$

Then, $R_{\varepsilon,\alpha}(\varphi) = \varepsilon^L(1/(1-\varepsilon\varphi)^L - 1)$. We also fix $V_{\varepsilon,\alpha}(r) = -r/(r+\varepsilon^L)^{1+\alpha}$, and hence

$$\frac{1}{Z_{\varepsilon,\alpha}}(\varphi) \sum_{k=0}^{\infty} \frac{\varphi^k (b_{\varepsilon,\alpha}(k) - d_{\varepsilon,\alpha}(k))}{g_{\varepsilon,\alpha}(k)!} = \frac{1}{\varepsilon}((1-\varepsilon\varphi)^{L+1} - (1-\varepsilon\varphi)) = \frac{1}{\varepsilon} P_\alpha(\varepsilon\varphi),$$

where P_α is a polynomial of degree $L+1$ in φ. Expanding, one finds that

$$b_{\varepsilon,\alpha}(k) - d_{\varepsilon,\alpha}(k) = \sum_{\ell=1}^{k} d_\ell \varepsilon^{\ell-1} g_{\varepsilon,\alpha}(k-\ell+1) \cdots g_{\varepsilon,\alpha}(k), \tag{3.8}$$

where

$$d_\ell = \begin{cases} 0, & \ell = 0, \\ -L, & \ell = 1, \\ \frac{(L+1)L\cdots(L+2-\ell)(-1)^\ell}{\ell!}, & \ell \geqslant 2, \end{cases} \tag{3.9}$$

and one notes that the sum in (3.8) is over at most $L+1$ terms since L is an integer and thus $d_\ell = 0$ for $\ell > L+1$. It thus only remains to compute the functions $g_{\varepsilon,\alpha}(k)$, a task considerably more involved than in the Euclidean case. Write $\log Z_{\varepsilon,\alpha}(\varphi) = \sum_{\ell=1}^{\infty} a_\ell \varphi^\ell$, with $a_\ell = L(L+1)\cdots(L+\ell-1)\varepsilon^{\ell+L}/\ell\ell!$. Expanding $Z_{\varepsilon,\alpha}(\varphi) = \sum_{k=0}^{\infty} t_k \varphi^k$, it holds that $g_{\varepsilon,\alpha}(k) = t_{k-1}/t_k$, with

$$t_k = \sum_{\lambda \vdash k} \frac{1}{|\lambda|!} \prod_{i=1}^{|\lambda|} a_{\lambda_i}, \tag{3.10}$$

where the summation is over the set $\mathcal{N}_k$ of all partitions $\lambda = (\lambda_1, \ldots, \lambda_{|\lambda|})$ of k, i.e. tuples of integers with $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_{|\lambda|} \geqslant 1$ such that $\sum \lambda_i = k$. We now have

Lemma 2. *There exist constants $c_{\varepsilon,\alpha}$, $C_{\varepsilon,\alpha}$ such that for all k,*

$$c_{\varepsilon,\alpha} \leqslant g_{\varepsilon,\alpha}(k) \leqslant C_{\varepsilon,\alpha}.$$

Due to Lemma 2 and (3.8) (recall L is an integer!), the functions $b_{\varepsilon,\alpha}(\cdot)$ and $d_{\varepsilon,\alpha}(\cdot)$ are also uniformly bounded, and Assumption B holds for the corresponding particle system.

We conclude this paragraph with

Proof of Lemma 2. Since for $\lambda \in \mathcal{N}_k$ it holds that $\sum \lambda_i = k$, we have that for $k \geqslant 2$,

$$g_{\varepsilon,\alpha}(k) = \frac{t_{k-1}}{t_k} = \frac{1}{\varepsilon} \frac{E_{k-1}(\frac{1}{|\lambda|!}\prod_{i=1}^{|\lambda|} Q_{\varepsilon,\alpha}(\lambda_i))}{E_k(\frac{1}{|\lambda|!}\prod_{i=1}^{|\lambda|} Q_{\varepsilon,\alpha}(\lambda_i))} \frac{N_{k-1}}{N_k},$$

where $N_k = |\mathcal{N}_k|$, E_k denotes the uniform measure over $\mathcal{N}_k$ and $Q_{\varepsilon,\alpha}(\cdot)$ is a rational function; hence,

$$\sup_{n \in \mathbb{N}} \frac{Q_{\varepsilon,\alpha}(n+1)}{Q_{\varepsilon,\alpha}(n)} < \infty, \quad \sup_{n \in \mathbb{N}} \frac{Q_{\varepsilon,\alpha}(n)}{Q_{\varepsilon,\alpha}(n+1)} < \infty.$$

Construct an injection I of $\mathcal{N}_{k-1}$ into a subset of $\mathcal{N}_k$ by increasing the first component $\lambda_1 \geqslant 1$ of λ by 1, i.e. $I(\lambda_1, \ldots, \lambda_{|\lambda|}) = (\lambda_1 + 1, \ldots, \lambda_{|\lambda|})$. In particular, I

leaves $|\lambda|$ unchanged. Then,

$$g_{\varepsilon,\alpha}(k) \leqslant \frac{1}{\varepsilon} \frac{\sum_{\mathcal{N}_{k-1}} \frac{1}{|\lambda|!} Q_{\varepsilon,\alpha}(\lambda_1) \prod_{i=2}^{|\lambda|} Q_{\varepsilon,\alpha}(\lambda_i)}{\sum_{\mathcal{N}_{k-1}} \frac{1}{|\lambda|!} Q_{\varepsilon,\alpha}(\lambda_1+1) \prod_{i=2}^{|\lambda|} Q_{\varepsilon,\alpha}(\lambda_i)} \leqslant \frac{1}{\varepsilon} \sup_{n \in \mathbb{N}} \frac{Q_{\varepsilon,\alpha}(n)}{Q_{\varepsilon,\alpha}(n+1)},$$

yielding the claimed upper bound on $g_{\varepsilon,\alpha}(\cdot)$. To see the complementary lower bound, for any $(\lambda_1, \ldots, \lambda_{|\lambda|}) \in \mathcal{N}_k$, set j_λ such that $\lambda_1 = \lambda_2 = \cdots = \lambda_{j_\lambda} > \lambda_{j_\lambda+1}$, with $j_\lambda = |\lambda|$ if $\lambda_1 = \cdots = \lambda_{|\lambda|}$. Construct a map J from $\mathcal{N}_k$ to $\mathcal{N}_{k-1}$ by reducing the λ_{j_λ} part by one, i.e.

$$J(\lambda_1, \ldots, \lambda_{|\lambda|}) = (\lambda_1, \ldots, \lambda_{j_\lambda} - 1, \lambda_{j_\lambda+1}, \ldots.$$

Note that the map J is two to one. Since $|J(\lambda)| \leqslant |\lambda|$, we have by an argument as above that

$$g_{\varepsilon,\alpha}(k) \geqslant \frac{1}{2\varepsilon \max(1, Q_{\varepsilon,\alpha}(1))} \inf_{n \in \mathbb{N}} \frac{Q_{\varepsilon,\alpha}(n-1)}{Q_{\varepsilon,\alpha}(n)},$$

completing the proof of the complementary lower bound. □

Remark. In the case of $\alpha = \frac{1}{3}$ (affine curve shortening [14]), one checks that $g_{\varepsilon,1/3}(k) \leqslant 1/\varepsilon$.

4. Proof of Theorem 2

As mentioned above, the strategy parallels that of the proof of the standard hydrodynamic limit for zero range processes, as described in [9], with some additional elements, adapted from [11], due to the presence of birth and death events. Set $v_{\rho(t,\cdot),N} := \otimes_{i \in T_N} v_{\rho(t,\frac{i}{N})}$. The main step in the proof of Theorem 2 consists of establishing

Proposition 2. *Let Assumptions* B *and* C *hold. Then*,

$$\limsup_{N \to \infty} \frac{1}{N} H(\mu_t^N | v_{\rho(t,\cdot),N}) = 0.$$

Indeed, let A denote the event

$$A = \left\{ \eta\colon \left| \frac{1}{N} \sum_{i \in T_N} \eta(i) G\left(\frac{i}{N}\right) - \int_{S^1} G(x) \rho(t,x)\, dx \right| > \delta \right\}.$$

Note that, by an inequality of Varadhan, see [11, p. 367],

$$\mu_t^N(A) \leqslant \frac{\frac{1}{N}\log 2 + \frac{1}{N}H(\mu_t^N | \nu_{\rho(t,\cdot),N})}{\frac{1}{N}\log(1 + 1/\nu_{\rho(t,\cdot),N})}.$$

In view of Proposition 2, it thus suffices to show that

$$\limsup_{N\to\infty} \frac{1}{N} \log \nu_{\rho(t,\cdot),N}(A) < 0. \tag{4.1}$$

The later estimate is a consequence of the product structure of $\nu_{\rho(t,\cdot),N}$ and of the existence of exponential moments as described in part (a) of Lemma 1. Indeed, the random variables $Z_i = \eta_i - \rho(t, i/N)$ are, under $\nu_{\rho(t,\cdot),N}$, independent, centered, and there exists a θ^* such that

$$\sup_{i,t<T} E_{\nu_{\rho(t,\cdot),N}}(e^{\theta^*|Z_i|}) < \infty\,.$$

Therefore, for any $G \in C(S^1)$, there exists a $C>0$ such that for all $a < a_0(G)$,

$$\sup_{i,t<T} E_{\nu_{\rho(t,\cdot),N}}(e^{aG(i/N)Z_i}). \tag{4.2}$$

Thus, by Chebycheff's inequality, we conclude that for every $a>0$,

$$\nu_{\rho(t,\cdot),N}(A) \leqslant e^{-Na\delta} E_{\nu_{\rho(t,\cdot),N}}(e^{a\sum_{i\in T_N}\eta(i)G(i/N) - \int_{S^1} G(x)\rho(t,x)\,dx}).$$

Approximating the last integral by a Riemann sum, we conclude that for every $\varepsilon>0$ we can find a $N_0(\varepsilon)$ such that for $N > N_0(\varepsilon)$,

$$\begin{aligned}\frac{1}{N}\log \nu_{\rho(t,\cdot),N}(A) &\leqslant -a\delta + \varepsilon + \frac{1}{N}\sum_{i\in T_N} \log E_{\nu_{\rho(t,\cdot),N}}(e^{aG(i/N)Z_i})\\ &\leqslant -a\delta + \varepsilon + Ca^2,\end{aligned}$$

where the second inequality is due to (4.2). Choosing $a < \delta/C$ one deduces (4.1), which concludes the proof of Theorem 2 modulo that we still need to prove Proposition 2.

The proof of Proposition 2 is provided in Section 4.2, after we first present in Section 4.1 a replacement lemma appropriate to our needs.

4.1. Replacement lemma

The main a priori estimate needed in our derivation is the following replacement lemma (compare with [11, Proposition 2.1]).

Proposition 3. *Let Assumptions* B *and* C *hold. Suppose* $h:\mathbb{N}\to\mathbb{R}$ *is sublinear at infinity*, *i.e.* $\limsup_{k\to\infty} h(k)/k = 0$. *For* $k\in\mathbb{N}$, *set*

$$\eta^k(i) = \frac{1}{2k+1}\sum_{|j-i|\leqslant k}\eta(j),\quad V_k(\eta)(i) = \left|\frac{1}{2k+1}\sum_{|i-j|\leqslant k} h(\eta(j)) - \tilde{h}(\eta^k(i))\right|.$$

Then,

$$\limsup_{k\to\infty}\ \limsup_{N\to\infty}\ E_{\mu^N}\left\{\frac{1}{N}\sum_{i=0}^{N-1}\int_0^T V_k(\eta_s)(i)\,ds\right\} = 0. \tag{4.3}$$

Proof. Following the proof of [11, Lemma 2.2], we have that

$$H(\mu_{t,N}|\nu_{1,N})\leqslant H(\mu_{s,N}|\nu_{1,N}) + \sup_{U\in C_b(T_N),U\geqslant 0}\log\left[\frac{\int S^N_{t-s}U(\eta)\nu_{1,N}(d\eta)}{\int U(\eta)\nu_{1,N}(d\eta)}\right]. \tag{4.4}$$

Recall that

$$\frac{d}{dt}\int S^N_t U(\eta)\nu_{1,N}(d\eta) = \sum_{i=0}^{N-1}\int\bigg[g(\eta(i))b(\eta(i)-1) - b(\eta(i)) + \frac{d(\eta(i)+1)}{g(\eta(i)+1)} - d(\eta(i))\bigg]U(\eta)\nu_{1,N}(d\eta). \tag{4.5}$$

Hence, using Assumption (B.3), (4.4) and the Gronwall lemma, one concludes that for any $0\leqslant s\leqslant t\leqslant T$,

$$H(\mu_{t,N}|\nu_{1,N})\leqslant H(\mu_{s,N}|\nu_{1,N}) + (t-s)C_2N. \tag{4.6}$$

We thus conclude that $f_{t,N} = d\mu_{t,N}/d\nu_{1,N}$ exists.

Define, for any f defined on T_N, the Dirichlet form $D_0[\cdot]$, as

$$D_0[f] = \frac{1}{4}\sum_{\substack{i\sim j\\(i,j)\in T_N\times T_N}} g(\eta(i))[\sqrt{f(\eta^{i,j})} - \sqrt{f(\eta)}]^2\nu_{1,N}(d\eta).$$

A repeat of the proof of [11, Lemma 2.3] yields

$$N^2D_0\left[\frac{1}{t}\int_0^t f_{s,N}\,ds\right]\leqslant\frac{1}{t}H(\mu_{0,N}|\nu_{1,N}) + C_3N. \tag{4.7}$$

Let

$$A_{N,C} = \left\{ f^N : \mathbb{N}^{T_N} \mapsto R_+ \mid \int f^N(\eta)\nu_{1,N}(d\eta) = 1, \right.$$
$$\left. D_0[f^N] \leqslant \frac{C}{N}, \int \eta(0) f^N(\eta)\nu_{1,N}(d\eta) \leqslant C, f^N \text{ is shift invariant} \right\}.$$

Copying the argument of [11, p. 370], it follows that Proposition 3 holds as soon as one can show that for any $C > 0$,

$$\limsup_{k\to\infty} \limsup_{N\to\infty} \sup_{f^N \in A_{N,C}} \int V_k(\eta)(0) f^N(\eta)\nu_{1,N}(d\eta) = 0. \tag{4.8}$$

Since on $A_{N,C}$ it holds that $\int \eta(0) f^N(\eta)\nu_{1,N}(d\eta) \leqslant C$, it follows that (4.8) holds as soon as for any $a > 0$,

$$\limsup_{k\to\infty} \limsup_{N\to\infty} \sup_{f^N \in A_{N,C}} \int f^N(\eta)[V_k(\eta)(0) - a\eta^k(0)]\nu_{1,N}(d\eta) \leqslant 0. \tag{4.9}$$

Note that due to Assumption (B.2), it holds that $E_{p_\alpha}(1_{X<k}) \to_{\alpha\to\infty} 0$ for any fixed k. Using this and the sublinear assumption on h, it follows that $\limsup_{\alpha\to\infty} |\tilde{h}(\alpha)|/\alpha = 0$. Using again the sublinearity of h, one concludes that

$$\limsup_{\eta^k(0)\to\infty} \frac{V_k(\eta)(0)}{\eta^k(0)} = 0. \tag{4.10}$$

and hence, (4.9) holds as soon as we show that for any constant $C' > 0$,

$$\limsup_{k\to\infty} \limsup_{N\to\infty} \sup_{f^N \in A_{N,C}} \int f^N(\eta) V_k(\eta)(0) \mathbf{1}_{\eta^k(0) < C'} \nu_{1,N}(d\eta) = 0. \tag{4.11}$$

To prove (4.11), we proceed by conditioning. Let $\nu_{1,k,N}$ (respectively, $\nu^c_{1,k,N}$) denote the restriction of $\nu_{1,N}$ to the (respectively, complement of the) box $B_k \coloneqq [-k, k]$ (we assume $N > 2k + 1$ such that B_k is identified as part of the torus T_N), and note that $\nu_{1,k,N} = \nu_{1,k}$ because $\nu_{1,N}$ is a product measure. Set

$$f^N_k(\xi) = \int f^N(\eta) \mathbf{1}_{\{\eta|_{B_k} = \xi\}} d\nu^c_{1,k,N}(d\eta)$$

and define the Dirichlet form D_k on functions $\zeta^k : \mathbb{N}^{B_k} \to \mathbb{R}$ by

$$D_k[\zeta^k] = \frac{1}{4} \sum_{j,j+1 \in B_k} \int g(\xi(j))(\sqrt{\zeta^k(\xi^{j,j+1})} - \sqrt{\zeta^k(\xi)})^2 \nu_{1,k}(d\xi)$$
$$+ \frac{1}{4} \sum_{j,j+1 \in B_k} \int g((\xi(j+1))(\sqrt{\zeta^k(\xi^{j+1,j})} - \sqrt{\zeta^k(\xi)})^2 \nu_{1,k}(d\xi)$$

then, as in [11, p. 372], using that V_k depends on η only through its restriction to B_k, it follows that

$$\sup_{f^N \in A_{N,C}} \int V_k(\eta)(0) f^N(\eta) \mathbf{1}_{\eta^k(0) < C'} \nu_{1,N}(d\eta) \leqslant \sup_{\zeta^k \in A^k_{N,C}} \int V_k(\eta)(0)\zeta^k(\eta)\mathbf{1}_{\eta^k(0) < C'} \nu_{1,N}(d\eta),$$

where

$$A^k_{N,C} = \left\{ \zeta^k \colon \zeta^k \geqslant 0, \int \zeta^k(\eta)\nu_{1,k}(d\eta) = 1, D_k[\zeta^k] \leqslant \frac{2k}{N^2} C, \int \eta(0)\zeta^k(\eta)\nu_{1,k}(d\eta) \leqslant C \right\}.$$

Consider $A^k_{N,C}$ as a set of densities, and hence identify it with a subset of $M_1(\mathbb{N}^{2k+1})$. Then, $A^k_{N,C}$ is compact under the weak topology of $M_1(\mathbb{N}^{2k+1})$, and the lower semicontinuity of $D_k[\cdot]$ yields that

$$\begin{aligned} &\limsup_{N\to\infty} \sup_{\zeta^k \in A^k_{N,C}} \int V_k(\eta)(0)\zeta^k(\eta)\mathbf{1}_{\zeta^k(0) < C'} \nu_{1,k}(d\eta) \\ &\quad \leqslant \sup_{\zeta^k \in A^k_0} \int V_k(\eta)(0)\zeta^k(\eta)\mathbf{1}_{\eta^k(0) < C'} \nu_{1,k}(d\eta) =: \mathscr{A}_k, \end{aligned}$$

where

$$A^k_0 = \left\{ \zeta^k \colon \zeta^k > 0, \int \zeta^k(\eta)\nu_{1,k}(d\eta) = 1, D_k[\zeta^k] = 0, \int \eta^k(0)\zeta^k(\eta)\nu_{1,k}(d\eta) \leqslant C \right\}.$$

We thus need to prove that $\limsup_{k\to\infty} \mathscr{A}_k = 0$. Toward this end, we do not use the argument in [11] but rather adapt [9, p. 89]. Indeed, let $\nu^j_{1,K}$ denote the law $\nu_{1,k}$ conditioned on $\eta^k(0) = j \cdot (2k+1)$. Then,

$$\mathscr{A}_k \leqslant \sup_{j \leqslant (2k+1)C'} \int V_k(\eta)(0)\nu^j_{1,k}(d\eta). \tag{4.12}$$

Noting (4.10) and repeating verbatim the equivalence of ensemble argument in [9, pp. 89–90], we conclude that $\limsup_{k\to\infty} \mathscr{A}_k = 0$, completing the proof of Proposition 3. □

4.2. *Relative entropy convergence: proof of Proposition 2*

We adopt the relative entropy method, as described in detail in [9, Chapter 6]. We emphasize in this presentation the ingredients which differ from the derivation there.

Set $\psi_N(t) := d\nu_{\rho(t,\cdot),N}/d\nu_{\alpha,N}$, $\alpha > 0$ arbitrary. Repeating the computation in [9, pp. 120–121], taking into account the birth–death rates, we conclude that

$$\begin{aligned}
\gamma_t &:= \frac{d}{dt} H(\mu_t^N | \nu_{\rho(t,\cdot),N}) \\
&\leqslant \sum_{x \in T_N} \int \bar{F}\Big(t, \frac{x}{N}\Big) \Big\{ g(\eta(x)) - \Phi\Big(\rho\Big(t, \frac{x}{n}\Big)\Big) \\
&\quad - \Phi'\Big(\rho\Big(t, \frac{x}{N}\Big)\Big)\Big(\eta(x) - \rho\Big(t, \frac{x}{N}\Big)\Big)\Big\} \nu_{1,N}(d\eta) \\
&\quad + \sum_{x \in T_N} \int \left[\frac{d(\eta(x)+1)}{g(\eta(x)+1)} \Phi\Big(\rho\Big(t, \frac{x}{N}\Big)\Big) - d(\eta(x)) \right] \nu_{1,N}(d\eta) \\
&\quad + \sum_{x \in T_N} \int \left[\frac{b(\eta(x)-1)}{\Phi(\rho(t, \frac{x}{N}))} g(\eta(x)) - b(\eta(x)) \right] \nu_{1,N}(d\eta) \\
&\quad - \sum_{x \in T_N} \int \Big(\eta(x) - \rho\Big(t, \frac{x}{N}\Big)\Big) \frac{\Phi'\Big(\rho\Big(t, \frac{x}{N}\Big)\Big)}{\Phi(\rho(t, \frac{x}{N}))} V\Big(\rho\Big(t, \frac{x}{N}\Big)\Big) \nu_{1,N}(d\eta) + o(N) \\
&=: \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV} + o(N),
\end{aligned}$$

where $\bar{F}(t, \frac{x}{N}) = \frac{\Delta\Phi(\rho(t, \frac{x}{N}))}{\Phi(\rho(t, \frac{x}{N}))}$, and where the $o(N)$ term is uniform in α in compacts. Note next that

$$\frac{1}{Z(\varphi)} \sum_{k=0}^{\infty} \frac{d(k+1)}{g(k+1)} \frac{\varphi^k}{g(k)!} = \frac{1}{\varphi} E_{\bar{p}_\varphi}(d)$$

and

$$\frac{1}{Z(\varphi)} \sum_{k=1}^{\infty} b(k-1) g(k) \frac{\varphi^k}{g(k)!} = \frac{1}{\varphi} E_{\bar{p}_\varphi}(b),$$

and thus

$$\left(\widetilde{\frac{d(\cdot+1)}{g(\cdot+1)}} \right)(a) = \frac{V_-(a)}{\Phi(a)}, \tag{4.13}$$

$$(\widetilde{b(\cdot-1)g(\cdot)})(a) = V_+(a)\Phi(a). \tag{4.14}$$

We next wish to replace functions depending on η by functions depending on η^k. Toward this end, note that by (4.6),

$$H(\mu_{t,N} | \nu_{1,N}) \leqslant C_3 N$$

and hence, for any bounded test function $\beta(x)$,

$$E_{\mu_{t,N}}\beta \leqslant \log E_{\nu_{1,N}}(e^{\beta}) + C_3 N.$$

Note next that for some $\gamma > 0$,

$$\limsup_{N\to\infty} \frac{1}{N} \log E_{\nu_{1,N}}\left(e^{\gamma \sum_{i\in T_N} \eta(i)}\right) = \log E_{\nu_{1,N}}(e^{\gamma\eta(0)}) < \infty.$$

Hence, by dominated convergence,

$$\limsup_{N\to\infty} \sup_{t\in[0,T]} E_{\mu_{t,N}} \frac{1}{N} \sum_{i\in T_N} \eta(i) < \infty. \tag{4.15}$$

In particular, for any smooth test function $\ell(x)$, for each fixed k,

$$\limsup_{N\to\infty} \sum_{x\in T^N} \int \ell\left(\frac{x}{N}\right)[\eta(x) - \eta^k(x)]\mu_{t,N}(d\eta) = 0 \tag{4.16}$$

with the convergence rate depending only on the modulus of continuity of $\ell(\cdot)$.

We next note that the functions $g(\cdot), \frac{d}{g}(\cdot), b(\cdot - 1)g(\cdot), b(\cdot)$ satisfy the assumptions of Proposition 2. Using the (uniform) space regularity of $\rho(t,\cdot)$, the smoothness (C^2 property) of $\Phi(\cdot)$ assured by Lemma 1[(b)], and summation by parts using (4.16), we conclude, using (4.13), that

$$\begin{aligned}
\gamma_t \leqslant & \sum_{x\in T_N} \int \bar{F}\left(t,\frac{x}{N}\right)\left\{\Phi(\eta^k(x)) - \Phi\left(\rho\left(t,\frac{x}{N}\right)\right)\right. \\
& \left. - \Phi'\left(\rho\left(t,\frac{x}{N}\right)\right)\left(\eta^k(x) - \rho\left(t,\frac{x}{N}\right)\right)\right\}\mu_{t,N}(d\eta) \\
& + \sum_{x\in T_N} \int V_-(\eta^k(x))\left[\frac{\Phi\left(\rho\left(t,\frac{x}{N}\right)\right)}{\Phi(\eta^k(x))} - 1\right]\mu_{t,N}(d\eta) \\
& + \sum_{x\in T} \int V_+(\eta^k(x))\left[\frac{\Phi(\eta^k(x))}{\Phi(\rho(t,\frac{x}{N}))} - 1\right]\mu_{t,N}(d\eta) \\
& - \sum_{x\in T_N} \int \left(\eta^k(x) - \rho\left(t,\frac{x}{N}\right)\right)\frac{\Phi'}{\Phi}\left(\rho\left(t,\frac{x}{N}\right)\right)V\left(\rho\left(t,\frac{x}{N}\right)\right)\mu_{t,N}(d\eta) \\
& + o(N),
\end{aligned} \tag{4.17}$$

where the error term in (4.17) is uniform in $t\in[0,1]$.

Rearranging the terms in (4.17), and setting $M(a,b) = \Phi(a) - \Phi(b) - \Phi'(b)(a-b)$, we get

$$
\begin{aligned}
\gamma_t \leqslant & \sum_{x \in T_N} \int \bar{F}\left(t, \frac{x}{N}\right) M\left(\eta^k(x), \rho\left(t, \frac{x}{N}\right)\right) \mu_{t,N}(d\eta) \\
& - \sum_{x \in T_N} \int \frac{V_-(\eta^k(x))}{\Phi(\eta^k(x))} M\left(\eta^k(x), \rho\left(t, \frac{x}{N}\right)\right) \mu_{t,N}(d\eta) \\
& + \sum_{x \in T_N} \int \frac{V_+(\eta^k(x))}{\Phi(\rho(t, \frac{x}{N}))} M\left(\eta^k(x), \rho\left(t, \frac{x}{N}\right)\right) \mu_{t,N}(d\eta) \\
& - \sum_{x \in T_N} \int \left[V_-(\eta^k(x)) - V_-\left(\rho\left(t, \frac{x}{N}\right)\right)\right] \left[\eta^k(x) - \rho\left(t, \frac{x}{N}\right)\right] \\
& \times \frac{\Phi'}{\Phi}\left(\rho\left(t, \frac{x}{N}\right)\right) \mu_{t,N}(d\eta) \\
& - \sum_{x \in T_N} \int \frac{V_-(\eta^k(x))}{\Phi(\eta^k(x))} \frac{\Phi'}{\Phi}\left(\rho\left(t, \frac{x}{N}\right)\right) \left[\eta^k(x) - \rho\left(t, \frac{x}{N}\right)\right] \\
& \times \left[\Phi\left(\rho\left(t, \frac{x}{N}\right)\right) - \Phi(\eta^k(x))\right] \mu_{t,N}(d\eta) \\
& - \sum_{x \in T_N} \int \frac{\left[V_+(\eta^k(x)) - V_+\left(\rho\left(t, \frac{x}{N}\right)\right)\right]}{\Phi(\rho(t, \frac{x}{N}))} \left(\Phi(\eta^k(x)) - \Phi\left(\rho\left(t, \frac{x}{N}\right)\right)\right) \mu_{t,N}(d\eta) \\
& + o(N) := \int \sum_{x \in T_N} \sum_{i=1}^{6} A_i(x, \eta) \mu_{t,N}(d\eta) + o(N),
\end{aligned} \tag{4.18}
$$

where again the error term is uniform in $t \in [0, T]$, and we have used (4.13) to assert that $\sup_x \frac{V_-}{\Phi}(x) < \infty$.

The proof of the following proposition follows the proof of [9, Proposition 6.1.6] and is therefore omitted. Note that introducing the supremum over t in the statement does not modify the proof due to the uniform bound on $\rho(t, x), t \in [0, T], x \in S^1$.

Proposition 4. *Let $G(\cdot, \cdot, \cdot) : [0, T] \times S^1 \times \mathbb{R}_+ \to \mathbb{R}_+$ be continuous, such that for some $C_0 > 0$*

(a) $\sup_{(t,u) \in [0,T] \times S^1} G(t, u, \lambda) \leqslant C_0 + C_0 \lambda, \quad \lambda \in \mathbb{R}_+$;
(b) $\sup_{\substack{(t,u) \in [0,T] \times S^1 \\ |\lambda - \rho(t,u)| < \delta}} G(t, u, \lambda) \leqslant C_0 \delta^2$.

Then, there exists a $\bar{\gamma}_0 = \bar{\gamma}_0(C_0)$ such that

$$
\limsup_{k \to \infty} \limsup_{N \to \infty} \sup_{t \in [0,T]} \frac{1}{N} \log E_{\nu_{\rho(t,\cdot),N}} \exp\left\{ \bar{\gamma}_0 \sum_{x \in T_N} G(t, x, \eta^k(x)) \right\} \leqslant 0.
$$

Equipped with Proposition 4, let us complete the proof of Proposition 2. Indeed, note that

$$\sum_{i=1}^{3} A_i(x,\eta) \leqslant C_{0,1} \left| M\left(\eta^k(x), \rho\left(t, \frac{x}{N}\right)\right) \right| \tag{4.19}$$

while

$$\sum_{i=4}^{6} A_i(x,\eta) \leqslant C_{0,2} \left| \eta^k(x) - \rho\left(t, \frac{x}{N}\right) \right| Q\left(\left| \eta^k(x) - \rho\left(t, \frac{x}{N}\right) \right| \right), \tag{4.20}$$

where Q is a smooth function, bounded by 1, with $Q(0) = 0$, and we used the fact that V_- is bounded which is assured by Assumption (B.3). Fixing $\bar{\gamma}_1$ small enough, and with a term $o(N)$ uniform in t,

$$\begin{aligned} \gamma_t - \gamma_0 \leqslant\; & o(N) + \frac{1}{\bar{\gamma}_1} \int_0^t \gamma_s \, ds \\ & + \frac{1}{\bar{\gamma}_1} \int_0^t ds \log E_{\nu_{\rho(t,\cdot),N}} \left[\exp\left\{ \bar{\gamma}_1 \sum_{x \in T_N} \sum_{i=1}^{6} A_i(x,\eta) \right\} \right]. \end{aligned}$$

Using Proposition 4 and (4.19), (4.20), it follows that

$$\limsup_{k \to \infty} \limsup_{N \to \infty} \sup_{t \in [0,T]} \frac{1}{N} \log E_{\nu_{\rho(t,\cdot),N}} \left[\exp\left\{ \bar{\gamma}_1 \sum_{x \in T_N} \sum_{i=1}^{6} A_i(x,\eta) \right\} \right] \leqslant 0$$

and thus, Gronwall's lemma yields that

$$\limsup_{N \to \infty} \sup_{t \in [0,T]} \frac{\gamma_t}{N} = 0. \qquad \square$$

5. Conclusions and future research

In this paper, we formulated certain stochastic approximations to planar shortening flows for convex curves. More precisely, we constructed a stochastic particle system whose profile defines an atomic measure on the unit circle such that the corresponding curve is a convex polygon. We then showed that the evolution of this polygonal curve converges (in the limit of a large number of particles) to curve evolution under the given curve-shortening flow.

We would like to suggest several possible research directions to extend these results. First of all, one can consider evolutions of non-convex curves. More precisely, it is known that for $\alpha = 1, \frac{1}{3}$ a smooth non-convex embedded curve becomes convex under the corresponding curve-shortening flow, and then converges to a point of appropriate "shape" (circular for $\alpha = 1$ [8], and elliptical for $\alpha = \frac{1}{3}$ [3]).

It would be quite interesting to see if one could extend our stochastic framework to non-convex curves in this setting.

Further, as alluded to above, our work here is partially motivated by the result that the uniform measure on the set of convex polygons of area bounded by 1 which encircle the origin and possesses vertices on the lattice $n^{-1}\mathbb{Z}^2$ satisfies a large deviation principle with rate function related to affine arc length [17]. Hence, we believe that natural (random) dynamics for these polygons should be related to evolution according to affine curve shortening. In our approach here, there does not seem to be anything special about the exponent $\alpha = \frac{1}{3}$. Thus, more research is necessary to see if one can indeed find "affine invariant" stochastic approximations to the affine curve-shortening evolution.

Appendix. Proofs of Proposition 1 and Theorem 1

We begin by recalling the following maximum principle, which is a straightforward adaptation to the periodic setting of [13, Theorem 12, p. 187]:

Lemma A.1. *Assume Φ, V satisfy Assumption* A *and let $m_1(\cdot), m_2(\cdot) \in C^{2+\beta}(S^1)$ satisfy $m_1 \leqslant m_2$. Let $\rho_i(t,x)$ satisfy $\rho_i(0,x) = m_i(x)$, $i = 1,2$, and $L\rho_1(t,x) \leqslant L\rho_2(t,x)$, for all $t \leqslant T$. Then, $\rho_1(t,x) \leqslant \rho_2(t,x)$ for $(t,x) \in [0,T] \times S^1$.*

The only issue preventing one from applying directly classical existence and uniqueness results for quasilinear parabolic equations is the fact that $\Phi'(\cdot)$ is not bounded away from 0 at infinity, and hence L is not a strictly parabolic operator. To circumvent this difficulty, assume $0 \leqslant \mu_1 \leqslant m(x) \leqslant \mu_2 < \infty$ for some constants $\mu_i, i = 1,2$, and let $\mu_i(t)$ satisfy the ODE

$$\frac{d\mu_i(t)}{dt} = V(\mu_i(t)), \quad \mu_i(0) = \mu_i. \tag{A.1}$$

Since V is Lipschitz, bounded and $V(0) \geqslant 0$, it holds that $0 < \mu_1(t) \leqslant \mu_2(t) \leqslant \mu_2 + ||V||t$ for all $t \geqslant 0$. An application of Lemma A.1 then yields that any solution $\rho(t,x)$ of (2.2) satisfies $0 < \mu_1(t) \leqslant \rho(t,x) \leqslant \mu_2(t)$. Fix $T < \infty$ and $\delta > 0$ such that $\delta < \min_{t \in [0,T]} \mu_1(t) < \mu_2 + ||V||T < 1/\delta$, and set Φ^δ be a smooth function with $\Phi^\delta(u) = \Phi(u)$ for $u \in [\delta, 1/\delta]$, such that $\min_{x \in \mathbb{R}_+} (\Phi^\delta)'(x) > 0$. Let L^δ denote the operator L of (2.1) with Φ^δ replacing Φ. By [10, Theorem 12.14], the equation $L^\delta \rho^\delta(t,x) = 0$, $\rho^\delta(0,x) = m(x)$ possesses a unique solution (the hypotheses of [10, Theorem 12.14] are checked to hold for the operator L^δ considered as an operator defined on $C^{1,2}([0,T] \times \mathbb{R})$, with the initial condition m extended by periodicity to $\mathbb{R}$, with the resulting unique solution being periodic and defining uniquely a periodic solution ρ^δ which then can be considered as defined on $[0,T] \times S^1$). By Lemma A.1 and the argument above, $\rho^\delta(t,x) \in [\delta, 1/\delta]$ for $t \leqslant T$. Hence, ρ^δ satisfies (2.2), establishing the claimed existence since $T > 0$ is arbitrary. The uniqueness follows by noting that any solution of (2.2) satisfies, by the above a priori bounds, that $\Phi(\rho(t,x)) = \Phi^{\delta}(\rho(t,x))$ for $t \leqslant T$, and hence is the (unique) solution of the equation $L^{\delta}\rho^{\delta}(t,x) = 0$.

Proof of Theorem 1. Fix $\gamma \in (0,1]$ and set $\bar{\rho}(t,x) = \rho(t,x)e^{\gamma t}$. A direct computation yields that

$$\begin{aligned} L^{\varepsilon}(\bar{\rho}) &= [\partial_{xx}\rho e^{\gamma t}(\Phi'(\rho) - \Phi'_{\varepsilon}(\bar{\rho}))] + [(e^{\gamma t}\partial_x \rho)^2 (e^{-\gamma t}\Phi''(\rho) - \Phi''_{\varepsilon}(\bar{\rho}))] \\ &\quad + [e^{\gamma t} V(\rho) - V_{\varepsilon}(\bar{\rho})] + \gamma\bar{\rho} \\ &=: I_1 + I_2 + I_3 + \gamma\bar{\rho}. \end{aligned} \tag{A.2}$$

We can find a constant $C = C(\delta) > 0$ independent of the values of γ and ε such that

$$\min_{(t,x)\in[0,T-\delta]\times S^1} \rho(t,x) \wedge \bar{\rho}(t,x) \geqslant \frac{1}{C}, \qquad \max_{(t,x)\in[0,T-\delta]\times S^1} \rho(t,x) \vee \bar{\rho}(t,x) \leqslant C,$$

$$\max_{(t,x)\in[0,T-\delta]\times S^1} [|\partial_{xx}\bar{\rho}(t,x)| + |\Phi'(\rho(t,x))| + |\Phi'_{\varepsilon}(\bar{\rho}(t,x))|] \leqslant C.$$

One therefore concludes the existence of a constant $C_1 = C_1(\delta)$ independent of ε or γ such that

$$\limsup_{\varepsilon\to 0} \max_{(t,x)\in(0,T-\delta]\times S^1} \frac{|I_1 + I_2 + I_3|}{s} \leqslant C_1 \gamma.$$

Setting $T_1 = (T-\delta) \wedge 1/2CC_1$, one concludes the existence of a function $\varepsilon_0(\cdot)$, depending on C and the rate of convergence of $\Phi'_{\varepsilon}, \Phi''_{\varepsilon}, V_{\varepsilon}$ only, such that for all $\varepsilon < \varepsilon_0(\gamma)$,

$$0 = L^{\varepsilon}\rho_{\varepsilon}(t,x) \leqslant L^{\varepsilon}\bar{\rho}(t,x), \quad (t,x) \in [0,T_1] \times S^1.$$

The maximum principle (Lemma A.1 above) then yields that for $\varepsilon < \varepsilon_0(\gamma)$, and $t \in [0,T_1]$, it holds that $\rho^{\varepsilon}(t,x) \leqslant e^{\gamma t}\rho(t,x)$. Repeating the argument with $\gamma \in [-1,0)$, and noting that T_1 does not depend on ε, the conclusion of the theorem follows on $[0,T_1] \times S^1$. The extension to $[0,T-\delta] \times S^1$ is immediate by noting that the constants C, C_1 do not depend on ε, and repeating the argument above $\lceil (T-\delta)/T_1 \rceil$ times. □

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