Hölder norms and the support theorem for diffusions

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Summary. One shows that the Stroock-Varadhan [S-V] support theorem is valid in α-Hölder norm. The central tool is an estimate of the probability that the Brownian motion has a large Hölder norm but a small uniform norm.

Contents :

Introduction

What is the probability that the Brownian motion oscillates rapidly conditionally on the fact that it is small in uniform norm? More precisely, what is the probability that the α-Hölder norm of the Brownian motion is large conditionally on the fact that its uniform norm (or more generally its β-Hölder norm with $\beta < \alpha$) is small?

This is the kind of question that naturally appears if one wants to extend the Stroock-Varadhan characterization of the support of the law of diffusion processes [S-V] to sharper topologies than the one induced by the uniform norm.

We deal with this question in §1 and show that those tails are much smaller than the gaussian tails one would get without the conditioning. This gives a family of examples where the conjecture (stated in [DG-E-...]) that two convex symmetric bodies are positively correlated (for gaussian measures) is true.

Our proofs are based on the Ciesielski isomorphism [C] (see [B-R] for other applications of this theorem) and on the correlation inequality. We give in appendix a proof which avoids these tools.

This enables us to control in §3 the probability that a Brownian stochastic integral oscillates rapidly conditionally on the fact that the Brownian motion is small in uniform norm. This is the tool to extend the Stroock-Varadhan support theorem to α-Hölder norms.

1. Conditional tails for oscillations of the Brownian motion

If x is a continuous real function on $[0,1]$, vanishing at zero, we define the sequence $(\,\xi_m(x)\,)_{m\geq 1}$ by the formula:

$$\xi_m(x) = \xi_{2^n+k}(x) = 2^{\frac{n}{2}} \left(2x\left(\frac{2k-1}{2^{n+1}}\right) - x\left(\frac{k}{2^n}\right) - x\left(\frac{k-1}{2^n}\right) \right) ,$$

for $n \geq 0$ and $k = 1, \ldots, 2^n$. Denote

$$\|x\|_0 = \sup_{0\leq t\leq 1} |x_t| \ , \tag{1.1}$$

$$\|x\|_\alpha = \sup_{0\leq s\neq t\leq 1} \frac{|x_t - x_s|}{|t-s|^\alpha} \ , \ \alpha \in]0,1] \, , \tag{1.2}$$

$$\|x\|'_\alpha = \sup_{m\geq 1} |m^{\alpha-\frac{1}{2}}\xi_m(x)| \ , \ \alpha \in \ [0,1] \, . \tag{1.3}$$

It is now classical that, for $\alpha \in]0,1[$, the norms $\|\cdot\|_\alpha$ and $\|\cdot\|_{\alpha'}$ are equivalent (see [C]):

$$(1.4)\qquad 2^{\alpha-1}\,\|x\|'_\alpha \le \|x\|_\alpha \le 2^{-\frac{1}{2}}\,k_\alpha\,\|x\|'_\alpha\,,\ \alpha\in\,]0,1[\,,$$

where

$$k_\alpha = \frac{2}{(2^\alpha-1)(2^{1-\alpha}-1)}\,,$$

and

$$(1.5)\qquad 2^{-2}\,\|x\|'_0 \le \|x\|_0\,.$$

Let w be a liniar Brownian motion started from zero. We want to estimate the probability that $\|w.\|_\alpha$ is large conditionally on the fact that $\|w.\|_\beta$ is small. We shall first tackle the same problem with the norms $\|\cdot\|'$.

(1.6) **Theorem.** *Let (r,R) be a couple of real positive numbers, $v = \left(\frac{R^b}{r^a}\right)^{\frac{1}{b-a}}$ and denote*

$$(1.7)\qquad \Lambda_{\alpha,\beta}(r,R) = \frac{\varphi(v)}{v} + \frac{1}{a}\,R^{-\frac{1}{a}}\int_v^\infty \varphi(t)\,t^{\frac{1}{a}-2}\,dt\,,$$

where $\varphi(t) = \dfrac{e^{-\frac{t^2}{2}}}{\sqrt{2\pi}}$, $a = \dfrac{1}{2}-\alpha$, $b = \dfrac{1}{2}-\beta$. Then,

$$(1.8)\qquad P(\,\|w.\|'_\alpha > R\,|\,\|w.\|'_\beta < r\,) \le \frac{1}{\int_0^v \varphi(t)\,dt}\,\Lambda_{\alpha,\beta}(r,R)\,;$$

$$(1.9)\qquad P(\,\|w.\|'_\alpha > R\,|\,\|w.\|_\beta \le r\,) \le \Lambda_{\alpha,\beta}(p_\beta r,R)\,,$$

where $p_\beta = 2^{1-\beta}$, if $\beta>0$ and $p_0 = 4$;

$$(1.10)\qquad P(\,\|w.\|_\alpha > R\,|\,\|w.\|_\beta \le r\,) \le \Lambda_{\alpha,\beta}(p_\beta r, 2^{\frac{1}{2}}k_\alpha^{-1}R)\,.$$

To prove the theorem we need the following:

(1.11) **Lemma.** *Let us denote $n_0 = \left[\left(\frac{R}{r}\right)^{\frac{1}{b-a}}\right]$. Then*

$$(1.12)\qquad \sum_{n\geq n_0+1}\int_{Rn^a}^{\infty}\varphi(t)\,dt \leq \Lambda_{\alpha,\beta}(r,R)\,.$$

Proof. By the classical bound:

$$\int_t^{\infty}\varphi(s)\,ds \leq \frac{\varphi(t)}{t} \equiv \psi(t)\,,\; t>0\,,$$

and the fact that ψ is decreasing, we get:

$$\sum_{n\geq n_0+1}\int_{Rn^a}^{\infty}\varphi(t)\,dt \leq \sum_{n\geq n_0+1}\psi(Rn^a) =$$

$$\psi(R(n_0+1)^a)+\sum_{n\geq n_0+2}\psi(Rn^a)\leq \psi(v)+\int_{n_0+1}^{\infty}\psi(Rt^a)\,dt =$$

$$\frac{\varphi(v)}{v}+\frac{1}{a}R^{-\frac{1}{a}}\int_{R(n_0+1)^a}^{\infty}\psi(t)\,t^{\frac{1}{a}-1}\,dt\,.$$

From this the conclusion follows.

q.e.d.

We make another essential observation. If C, C' are two symmetric convex sets in $\mathbb{R}^d$, a general conjecture stated in [DG-E-...] predicts that they are positively correlated for the canonical Gaussian measure γ_d, that is,

$$(1.13)\qquad \gamma_d(C\cap C')\geq \gamma_d(C)\,\gamma_d(C')\,.$$

This is true for $d=2$ (see [P]), and for arbitrary d provided C' is a symmetric strip (see [Sc] or [Si]). The general case is stil open.

Proof of the Theorem (1.6).

Proof of (1.8). We note that $g_n=\xi_n(w)$ is a sequence of independent identically distributed standard Gaussian random variables. Then,

$$P(\,\|w.\|'_{\alpha}>R\,|\,\|w.\|'_{\beta}<r\,)=P(\sup_{n\geq 1}|n^{-a}g_n|>R\,|\,\sup_{m\geq 1}|m^{-b}g_m|<r\,)=$$

$$\frac{P(\cup_{n\geq 1}(\,|g_n|>Rn^a)\cap\cap_{m\geq 1}(\,|g_m|<rm^b))}{P(\cap_{m\geq 1}|g_m|<rm^b)}\leq$$

$$\frac{\sum_{n\geq 1} P((Rn^a < |g_n| < rn^b) \cap \cap_{m\geq 1, m\neq n}(|g_m| < rm^b))}{\prod_{m\geq 1} P(|g_m| < rm^b)} =$$

$$\sum_{n\geq 1} \frac{P(Rn^a < |g_n| < rn^b)}{P(|g_n| < rn^b)} \cdot 1_{(Rn^a < rn^b)} = \sum_{n\geq 1} \frac{2\int_{Rn^a}^{rn^b} \frac{e^{-\frac{s^2}{2}}}{\sqrt{2\pi}}\, ds}{2\int_0^{rn^b} \frac{e^{-\frac{s^2}{2}}}{\sqrt{2\pi}}\, ds} \cdot 1_{(Rn^a < rn^b)} =$$

$$\sum_{n\geq n_0+1} \frac{\displaystyle\int_{Rn^a}^{rn^b} \varphi(t)\, dt}{\int_0^{rn^b} \varphi(t)\, dt} \leq \frac{1}{\int_0^v \varphi(t)\, dt} \sum_{n\geq n_0+1} \int_{Rn^a}^{\infty} \varphi(t)\, dt\,.$$

Clearly $rn^b \geq r(n_0+1)^b \geq v$ so the last inequality is true. Then (1.8) is a consequence of the Lemma (1.11).

Proof of (1.9). We can write again

$$P(\|w.\|'_\alpha > R \,|\, \|w.\|_\beta \leq r) = P(\sup_{n\geq 1} |n^{-a} g_n| > R \,|\, \|w.\|_\beta \leq r) =$$

$$P(\cup_{n\geq 1}(|g_n| > Rn^a) \,|\, \|w.\|_\beta \leq r) \leq \sum_{n\geq 1} P(|g_n| > Rn^a \,|\, \|w.\|_\beta \leq r)\,.$$

But for $\|w.\|_\beta \leq r$, by (1.4) or (1.5) we get

$$|g_n| \leq 2^{1-\beta} rn^b \ , \text{ if } \ \beta > 0$$

or

$$|g_n| \leq 4rn^{\frac{1}{2}} \ , \text{ if } \ \beta = 0\,.$$

So, the preceding sum is taken over all integer $n \geq 1$ such that $2^{1-\beta} rn^b \geq Rn^a$, if $\beta > 0$, or $4rn^{\frac{1}{2}} \geq Rn^a$, if $\beta = 0$, that is,

$$n \geq 2^{\frac{\beta-1}{\alpha-\beta}} \left(\frac{R}{r}\right)^{\frac{1}{b-a}} \ \text{ or } \ n \geq 2^{-\frac{2}{\alpha}} \left(\frac{R}{r}\right)^{\frac{1}{\alpha}}\,.$$

On the other hand, $g_n = \xi_n(w)$ is a linear form on the Wiener space. By the correlation inequality with one of sets a symmetric strip and by a simple finite dimensional approximation (see also [S-Z]), we obtain

$$P(|g_n| > Rn^a \,,\, \|w.\|_\beta \leq r) \geq P(|g_n| > Rn^a)\, P(\|w.\|_\beta \leq r)$$

or

$$P(\,|g_n| > Rn^a \,|\, \|w.\|_\beta \le r\,) \le P(\,|g_n| > Rn^a\,)\,.$$

Therefore

$$P(\,\|w.\|'_\alpha > R \,|\, \|w.\|_\beta \le r\,) \le \sum_{n\ge n_1+1} P(\,|g_n| > Rn^a\,) = \sum_{n\ge n_1+1} \int_{Rn^a}^{\infty} \varphi(t)\,dt\,,$$

where $n_1 = \left[p_\beta^{\frac{1}{b-a}} \, \left(\frac{R}{r}\right)^{\frac{1}{b-a}} \right]$. By the Lemma (1.11) we get (1.9).

Proof of (1.10). It is a consequence of (1.4) or (1.5) and (1.9).

q.e.d.

We shall estimate $\Lambda_{\alpha,\beta}(r,R)$:

(1.14) **Lemma.** *With the notations of the* Theorem (1.6), *there exists a polynomial function* Ψ_a, *increasing on* $]0,\infty[$, *such that*

$$\Lambda_{\alpha,\beta}(r,R) \le \frac{\varphi(v)}{v} \left(1 + \frac{1}{a}\, R^{-\frac{1}{a}}\, v^{\frac{1}{a}-2}\, \Psi_a\left(\frac{1}{v}\right)\right)\,. \tag{1.15}$$

Proof. We shall simply give an upper bound for $\int_v^\infty \varphi(t)\, t^{\frac{1}{a}-2}\, dt$. Noting that $\varphi'(t) = -t\,\varphi(t)$ and integrating by parts, we get

$$\int_v^\infty \varphi(t)\, t^{\frac{1}{a}-2}\, dt = -\int_v^\infty \varphi'(t)\, t^{\frac{1}{a}-3}\, dt = \varphi(v)\, v^{\frac{1}{a}-3} +$$

$$\left(\frac{1}{a} - 3\right) \int_v^\infty \varphi(t)\, t^{\frac{1}{a}-4}\, dt\,.$$

If $a \ge \frac{1}{3}$,

$$\int_v^\infty \varphi(t)\, t^{\frac{1}{a}-2}\, dt \le \varphi(v)\, v^{\frac{1}{a}-3}\,,$$

which gives (1.14) with $\Psi_a(x) \equiv 1$. If $a < \frac{1}{3}$, similarly,

$$\int_v^\infty \varphi(t)\, t^{\frac{1}{a}-4}\, dt = \varphi(v)\, v^{\frac{1}{a}-5} + \left(\frac{1}{a} - 5\right) \int_v^\infty \varphi(t)\, t^{\frac{1}{a}-6}\, dt\,.$$

So, if $\frac{1}{5} \le a < \frac{1}{3}$,

$$\int_v^\infty \varphi(t)\, t^{\frac{1}{a}-2}\, dt \le \varphi(v)\, v^{\frac{1}{a}-3} + \left(\frac{1}{a} - 3\right)\, \varphi(v)\, v^{\frac{1}{a}-5}\,,$$

which is exactly (1.14) with $p = 1$ in the following expression:

$$\Psi_a(x) = 1 + \left(\frac{1}{a} - 3\right) x^2 + \ldots + \left(\frac{1}{a} - 3\right)\left(\frac{1}{a} - 5\right)\ldots\left(\frac{1}{a} - 2p - 1\right) x^{2p} .$$

Repeating the same reasoning the result is easily obtained for any p and any a such that $\frac{1}{2p+3} \leq a < \frac{1}{2p+1}$. Ψ_a has positive coefficients, it is therefore increasing on $]0, \infty[$.

q.e.d.

Combining this result with the Theorem (1.6) we obtain the following:

(1.16) **Corollary.** *Let (R, r) be such that $v \geq \varepsilon > 0$. Then,*

(1.17) $$P(\, \|w.\|'_\alpha > R \,|\, \|w.\|'_\beta < r\,) \leq c(\varepsilon)\,\frac{\varphi(v)}{\varepsilon}\left(1 + \Psi_a\left(\frac{1}{\varepsilon}\right)\left(\frac{R^\beta}{r^\alpha}\right)^{\frac{2}{\alpha-\beta}}\right) ;$$

(1.18) $$P(\, \|w.\|'_\alpha > R \,|\, \|w.\|_\beta < r\,) \leq \frac{\varphi(q_\beta v)}{q_\beta \varepsilon}\left(1 + q_\beta^{\frac{1}{a}-2}\Psi_a\left(\frac{1}{\varepsilon}\right)\left(\frac{R^\beta}{r^\alpha}\right)^{\frac{2}{\alpha-\beta}}\right) ;$$

(1.19) $$P(\, \|w.\|_\alpha > R \,|\, \|w.\|_\beta < r\,) \leq \frac{\varphi(c_{\alpha,\beta} v)}{c_{\alpha,\beta}\varepsilon}(1 + (2^{\frac{1}{2}} k_\alpha^{-1})^{-\frac{1}{a}} c_{\alpha,\beta}^{\frac{1}{a}-2} \cdot$$

$$\cdot \qquad \Psi_a\left(\frac{1}{\varepsilon}\right)\left(\frac{R^\beta}{r^\alpha}\right)^{\frac{2}{\alpha-\beta}}) .$$

Here $q_\beta = p_\beta^{-\frac{a}{b-a}}$, $c_{\alpha,\beta} = q_\beta\,(2^{\frac{1}{2}} k_\alpha^{-1})^{\frac{b}{b-a}}$ and $c(\varepsilon) = \dfrac{1}{\int_0^\varepsilon \varphi(t)\,dt}$. Note that if $\varepsilon \to \infty$ then $c(\varepsilon) \to 2$ and $\Psi_a(\frac{1}{\varepsilon}) \to 1$.

We prove now a stronger result:

(1.20) **Theorem.** *Let $\alpha\,, \beta$ be two real numbers such that $0 \leq \beta < \alpha < \frac{1}{2}$. There exists a positive number $u_{\alpha,\beta} = \frac{1-2\alpha}{1-2\beta}$, such that, for every $u \in [0, u_{\alpha,\beta}[$, there exists $M_0(\alpha, \beta, u)$ and positive constants $k_i(\alpha, \beta, u)\,,\ i = 1, 2$, such that, for every $M \geq M_0$,*

$$(1.21) \qquad \sup_{0<\delta\leq 1} P(\, \|w.\|_\alpha > M\delta^u \,|\, \|w.\|_\beta < \delta) \leq k_1 M^{\frac{2\beta}{\alpha-\beta}} \exp\left(-k_2\, M^{\frac{1-2\beta}{\alpha-\beta}}\right) .$$

Proof. First of all we take in the Corollary (1.16), $R = M\delta^u$ and $r = \delta$. So, for every $\delta \in]0,1]$,

$$P(\, \|w.\|_\alpha > M\delta^u \,|\, \|w.\|_\beta < \delta) \leq c_{\alpha,\beta}\, M^{\frac{1-2b}{b-a}}\, \delta^{\frac{u(1-2b)-(1-2a)}{b-a}} \cdot$$

$$\exp\left(-c'_{\alpha,\beta}\, M^{\frac{2b}{b-a}} \delta^{2\,\frac{ub-a}{b-a}}\right) .$$

It is clear that, when $M \geq \left(\frac{2}{c'_{\alpha,\beta}} \cdot \frac{u(1-2b)-(1-2a)}{b-a} \cdot \frac{b-a}{a-ub}\right)^{\frac{b-a}{2b}}$ the right hand side of the last inequality is an increasing function of δ, for $\delta \in]0,1]$. So,

$$\sup_{0<\delta\leq 1} P(\, \|w.\|_\alpha > M\delta^u \,|\, \|w.\|_\beta < \delta) \leq c_{\alpha,\beta}\, M^{\frac{1-2b}{b-a}} \exp\left(-c'_{\alpha,\beta}\, M^{\frac{2b}{b-a}}\right) ,$$

namely the conclusion.

q.e.d.

2. Hölder balls of different exponent positively correlated

We show here that the conjecture on the correlation inequality is true for Hölder balls. We denote $B_\alpha(\rho) = \{\, \|w.\|_\alpha \leq \rho \,\}$ and $B'_\alpha(\rho) = \{\, \|w.\|'_\alpha \leq \rho \,\}$.

(2.1) **Theorem.** *If R is sufficient large and if r is fixed, then $B_\alpha(R)$ and $B_\beta(r)$ are positively correlated.*

Proof. We proved in Corollary (1.16), for example when $r = 1$, that, for large R,

$$(2.2) \qquad P(\, B_\alpha(R)^c \,|\, B_\beta(1)\,) \leq c_{\alpha,\beta} \exp\left(-c'_{\alpha,\beta}\, R^{\frac{1-2\beta}{\alpha-\beta}}\right) ,$$

for every $0 \leq \beta < \alpha < \frac{1}{2}$. We can compare this estimate with the classical gaussian estimate, for large R,

$$(2.3) \qquad P(\, \|w.\|_\alpha > R\,) \leq \exp\left(-c_\alpha\, R^2\right)$$

(see [BA-Le] or [B-BA-K] for other consequences of this inequality).

By large deviations principle we obtain in fact,

$$P(B_\alpha(R)^c) \sim e^{-c_\alpha R^2} ,$$

provided R is sufficiently large. Therefore, by (2.2), for large R,

$$P(B_\alpha(R) \mid B_\beta(1)) \geq P(B_\alpha(R)) . \tag{2.4}$$

So, in this particular case, the general conjecture is valid: the two symmetric convex sets $B_\alpha(R)$ and $B_\beta(1)$ are positively correlated, for large R.

q.e.d.

Remark. We can also show that, for any $R, r > 0$, the pairs of balls $(B'_\alpha(R), B'_\beta(r))$ and $(B'_\alpha(R), B_\beta(r))$ are positively correlated. Indeed, by (1.3),

$$B'_\alpha(R) = \cap_{m\geq 1} \left(|g_m| \leq R m^{\frac{1}{2}-\alpha} \right) = \cap_{m\geq 1} S_m ,$$

so, it is an intersection of independent symmetric strips. Then, with the same argument as in the proof of (1.9), we get, for any convex symmetric C,

$$P(C \cap B'_\alpha(R)) = P\left(C \cap \cap_{m\geq 1} S_m\right) \geq$$

$$P\left(C \cap \cap_{m\geq 2} S_m\right) P(S_1) \geq \ldots \geq P(C) \prod_{m\geq 1} P(S_m) =$$

$$P(C)\, P\left(\cap_{m\geq 1} S_m\right) = P(C)\, P(B'_\alpha(R)) .$$

Here we used the independence of S_m. The conclusion is obtained taking $C = B'_\beta(r)$ or $C = B_\beta(r)$.

3. Conditional tails for oscillations of stochastic integrals

We shall estimate the Hölder norm of some stochastic integrals. Let $X_j(t,x)$, $j = 1, \ldots, m$, $X_0(t,x)$ be smooth vector fields on $\mathbb{R}^{d+1}$ and denote $(B^1, \ldots, B^m)$ a m-dimensional Brownian motion. Let P_x be the law of the diffusion (x_t), the solution of the Stratonovich equation

$$dx_t = \sum_{j=1}^{m} X_j(t, x_t) \circ dB^j_t + X_0(t, x_t)\, dt \, , \; x_0 = x \, . \tag{3.1}$$

Let us introduce the following class of stochastic processes:

(3.2) **Definition.** *For* $\alpha\,,\,\beta\,\in\,[0,\frac{1}{2}[$ *and* $u\,\in\,[0,1]$, *we shall denote by* $\mathcal{M}_u^{\alpha\,,\beta}$ *the set of stochastic processes* Y, *such that*

$$\lim_{M\uparrow\infty}\sup_{0<\delta\le 1} P(\,\|Y.\|_\alpha > M\delta^u \mid \|B.\|_\beta < \delta\,) = 0\,. \tag{3.3}$$

Here and elsewhere $\|B.\|_\alpha = \max_{1\le i\le m}\|B.^i\|_\alpha$. We collect our results in the following:

(3.4) **Lemma.** *Let* $f : \mathbb{R}^d \longrightarrow \mathbb{R}$ *be a smooth function and, for* $i, j \in \{1,\ldots,r\}$, *denote*

$$\eta_t^{ij} = \frac{1}{2}\int_0^t (B_s^i\,dB_s^j - B_s^j\,dB_s^i)\,,\ \xi_t^{ij} = \int_0^t B_s^i \circ dB_s^j\,. \tag{3.5}$$

Then,

(i) $\quad B.^i \in \mathcal{M}_u^{\alpha\,,\beta}$, *for* $0\le\beta<\alpha<\frac{1}{2}$ *and* $u\in\left[0,\frac{1-2\alpha}{1-2\beta}\right[$.

(ii) $\quad \eta.^{ij} \in \mathcal{M}_u^{\alpha\,,0}$, *for* $\alpha\in[0,\frac{1}{2}[$ *and* $u\in[0,1]$.

(iii) $\quad \xi.^{ij} \in \mathcal{M}_u^{\alpha\,,0}$, *for* $\alpha\in[0,\frac{1}{2}[$ *and* $u\in[0,1]$.

(iv) $\quad \int_0^\cdot f(x_s)\,d\xi_s^{ij} \in \mathcal{M}_u^{\alpha\,,0}$, *for* $\alpha\in[0,\frac{1}{2}[$ *and* $u\in[0,1]$.

(v) $\quad \int_0^\cdot f(x_s)\circ dB_s^i \in \mathcal{M}_u^{\alpha\,,0}$, *for* $\alpha\in[\,0\,,\,\frac{1}{2}[$ *and* $u\in[0,1-2\alpha[$.

Proof. Clearly, (i) is proved in the Theorem (1.20).

(ii) We proceed as in [S-V]. There exists a one dimensional Brownian motion w, such that, when $i\neq j$,

$$\eta_t^{ij} = w(a(t))\,,\ a(t) = \frac{1}{4}\int_0^t ((B_s^i)^2 + (B_s^j)^2)\,ds\,,$$

where w is independent of the process $(B_t^i)^2 + (B_t^j)^2$ and so, independent of $\|B.\|_0$. There exists a positive constant c, such that $\|a\|_0\,,\,\|a\|_1$ are bounded by $c\,\|B.\|_0$. Then we can write

$$P(\,\|\eta.^{ij}\|_\alpha > M\delta^u \,|\, \|B.\|_0 < \delta\,) =$$

$$P(\,\|B.\|_0 < \delta\,)^{-1} \cdot P(\,\|w(a(\cdot))\|_\alpha > M\delta^u\,,\, \|B.\|_0 < \delta\,)\,.$$

If z is α-Hölder, $\tilde z$ is β-Hölder then $z \circ \tilde z$ is $\alpha\beta$-Hölder and

$$\|z \circ \tilde z\|_{\alpha\beta} \le \|z\|_{\alpha\,,\,\|\tilde z\|_0} \cdot \|\tilde z\|_\beta^\alpha\,.$$

Here and elsewhere $\|\cdot\|_{\alpha,T}$ denotes the Hölder norm on $[0,T\,]$.

So,

$$\|w(a(\cdot))\|_\alpha \le \|w\|_{\alpha\,,\,\|a\|_0} \cdot \|a\|_1^\alpha\,.$$

Therefore

$$P(\,\|w(a(\cdot))\|_\alpha > M\delta^u\,,\, \|B.\|_0 < \delta\,) \le$$

$$P(\,\|w\|_{\alpha\,,\,c\,\|B.\|_0^2}\, c\,\|B.\|_0^{2\alpha} > M\delta^u\,,\, \|B.\|_0 < \delta\,)\,.$$

A scaling in Hölder norm shows that $\|w\|_{\alpha,\tau^2}$ and $\tau^{1-2\alpha}\,\|w\|_{\alpha,1}$ have the same law. Then we can write

$$P(\,\|\eta.^{ij}\|_\alpha > M\delta^u \,|\, \|B.\|_0 < \delta\,) \le$$

$$P(\,\|B.\|_0 < \delta\,)^{-1} \cdot P(\,\|w.\|_\alpha\, c\,\|B.\|_0^{1-2\alpha}\|B.\|_0^{2\alpha} > M\delta^u\,,\, \|B.\|_0 < \delta\,)\,.$$

Finally,

$$P(\,\|\eta.^{ij}\|_\alpha > M\delta^u \,|\, \|B.\|_0 < \delta\,) \le P(\,\|w.\|_\alpha\, c\,\delta > M\delta^u) \le \exp\left(-\frac{c_\alpha\, M^2}{\delta^{2(1-u)}}\right)\,,$$

by the independence of w and $\|B.\|_0$, and by the gaussian inequality (2.3).

(iii) We note another trivial inequality: if $z\,,\,\tilde z$ are α-Hölder then $z\,\tilde z$ is α-Hölder and

$$\|z\,\tilde z\|_\alpha \le \|z\|_\alpha\,\|\tilde z\|_0 + \|z\|_0\,\|\tilde z\|_\alpha\,.$$

In particular

$$\|B.^i B.^j\|_\alpha \le 2\,\|B.\|_0\,\|B.\|_\alpha\,.$$

But

$$P(\,\|B.\|_0\,\|B.\|_\alpha > M\delta^u \,|\, \|B.\|_0 < \delta\,) = P(\,\|B.\|_\alpha > M\delta^{u-1} \,|\, \|B.\|_0 < \delta\,)\,.$$

The conclusion follows at once from (i), (ii) and

$$\|\xi_\cdot^{ij}\|_\alpha \le \|\eta_\cdot^{ij}\|_\alpha + \frac{1}{2}\,\|B_\cdot^i B_\cdot^j\|_\alpha \le \|\eta_\cdot^{ij}\|_\alpha + \|B_\cdot\|_0\,\|B_\cdot\|_\alpha\,.$$

(iv) We apply Ito's formula several times (using the usual convention that repeated indices are summed):

$$\int_0^t f(x_s)\,d\xi_s^{ij} = f(x_t)\,\xi_t^{ij} - \int_0^t f_l(x_s)\,X_k^l(x_s)\,\xi_s^{ij}\,dB_s^k -$$

$$\int_0^t (L_s f)\,(x_s)\,\xi_s^{ij}\,ds - \int_0^t f_l(x_s)\,X_j^l(x_s)\,B_s^i\,ds = I_1 + I_2 + I_3 + I_4\,.$$

Here L_t is the generator of the diffusion (x_t) and X_j^l denotes the l component of X_j. It is sufficient to verify (iv) for each I_i, $i = 1,2,3,4$. We readily see that

(a) $$I_3\,,\ I_4 \in \mathcal{M}_u^{\alpha\,,0}\,,$$

because

$$\|I_4\| \le c\,\|B_\cdot\|_0 \quad\text{and}\quad \|I_3\| \le c\,\|\xi_\cdot^{ij}\|_0\,,$$

so, we consider only I_1 and I_2.

Firstly,

$$I_1 = f(x)\,\xi_t^{ij} + \left(\int_0^t (L_s f)\,(x_s)\,ds\right)\,\xi_t^{ij} + \left(\int_0^t f_l(x_s)\,X_k^l(x_s)\,\xi_s^{ij}\,dB_s^k\right)\,\xi_t^{ij} =$$

$$I_{10} + I_{11} + I_{12}\,.$$

Again

(b) $$I_{10}\,,\ I_{11} \in \mathcal{M}_u^{\alpha\,,0}\,,$$

because

$$\|I_{10}\|_\alpha = c\,\|\xi_\cdot^{ij}\|_\alpha\,,\ \|I_{11}\|_\alpha \le c\,\|\xi_\cdot^{ij}\|_\alpha\,.$$

Setting $\alpha_k = -f_l\,X_k^l$, $\alpha_{k,m} = \frac{\partial \alpha_k}{\partial x^m}$ we can write

$$I_{12} = -\alpha_k(x_t)\,B_t^k\,\xi_t^{ij} - \left(\int_0^t B_s^k (L_s\alpha_k)\,(x_s)\,ds\right)\,\xi_t^{ij} -$$

$$\left(\int_0^t B_s^k\,\alpha_{k,m}(x_s)X_n^m(x_s)\,dB_s^n\right)\,\xi_t^{ij} + \left(\int_0^t (\alpha_{k,p})^2(x_s)\,(X_k^p)^2(x_s)\,ds\right)\,\xi_t^{ij} =$$

$$I_{121} + I_{122} + I_{123} + I_{124}\,.$$

There is no problem to see that

$$\|I_{122}\|_\alpha \le c\,\|B.\|_0\,\|\xi.^{ij}\|_\alpha \ \text{ and } \ \|I_{124}\|_\alpha \le c\,\|\xi.^{ij}\|_\alpha$$

and so,

(c) $$I_{122}\,,\,I_{124} \in \mathcal{M}_u^{\alpha\,,0}\,.$$

There exists a one-dimensional Brownian motion w, such that

$$I_{123} = w(a(t))\,\xi_t^{ij}\ ,\ a(t) = \int_0^t (\alpha_{k,m}\alpha_{k',m'}a^{mm'})(x_s)B_s^k B_s^{k'}\,ds\,,$$

where $a^{ij} = \sum_{k=1}^m X_k^i\,X_k^j$. We obtain

$$P(\,\|I_{123}\|_\alpha > M\delta^u\,|\,\|B.\|_0 < \delta\,) \le P(\,\|\xi.^{ij}\|_\alpha > M^{\frac{1}{2}}\delta\,|\,\|B.\|_0 < \delta\,) +$$

$$P(\,\|w.\|_{\alpha\,,\,c\,\|B.\|_0^2}\,c\,\|B.\|_0^{2\alpha}\,\|\xi.^{ij}\|_\alpha > M\delta^u\,,\,\|\xi.^{ij}\|_\alpha \le M^{\frac{1}{2}}\delta\,|\,\|B.\|_0 < \delta\,)\,.$$

By (iii), we have to consider only the second term:

$$P(\,\|w.\|_\alpha > c\,M^{\frac{1}{2}}\delta^{u-2}) \cdot P(\,\|B.\|_0 < \delta\,)^{-1} \le \exp\left(-\frac{c_\alpha\,M}{\delta^{2(2-u)}} + \frac{c}{\delta^2}\right)\,.$$

This yields

(d) $$I_{123} \in \mathcal{M}_u^{\alpha\,,0}\,.$$

Then,

$$I_2 = \int_0^t \alpha_k(x_s)\,\xi_s^{ij}\,dB_s^k =$$

$$\alpha_k(x_t)\,\xi_t^{ij}\,B_t^k - \int_0^t \alpha_{k,l}(x_s)\,(X_m^l)(x_s)\,\xi_s^{ij}\,B_s^k\,dB_s^m - \int_0^t (L_s\alpha_k)\,(x_s)\,\xi_s^{ij}\,B_s^k\,ds$$

$$-\int_0^t \alpha_k(x_s)\,B_s^k\,d\xi_s^{ij} - \int_0^t \alpha_j(x_s)\,B_s^i\,ds - \int_0^t \xi_s^{ij}\,\alpha_{k,l}(x_s)\,X_m^l(x_s)\,\delta^{km}\,ds -$$

$$\int_0^t B^k_s\,\alpha_{k,l}(x_s)\,X^l_j(x_s)\,B^i_s\,ds = J_1+\cdots+J_7\,.$$

Clearly,

(e) $$I_{121}+J_1 = 0$$

and

$$\|J_3\|_\alpha \le c\,\|B.\|_0\,\|\xi.^{ij}\|_0\ ,\ \|J_5\|_\alpha \le c\,\|B.\|_0\,,$$
$$\|J_6\|_\alpha \le c\,\|\xi.^{ij}\|_0\ ,\ \|J_7\|_\alpha \le c\,\|B.\|_0^2\,.$$

So,

(f) $$J_3\,,\ J_5\,,\ J_6\,,\ J_7 \in \mathcal{M}^{\alpha\,,\,0}_u\,.$$

By the same reasoning,

$$J_2 = w(a(t))\ ,\ a(t) = \int_0^t (\xi^{ij}_s)^2(\alpha_{k,l}\,\alpha_{k',l'}\,a^{ll'})(x_s)\,B^k_s\,B^{k'}_s\,ds\,,$$

so, it suffices to estimate

$$P(\,\|J_2\|_\alpha > M\delta^u\,,\,\|\xi.^{ij}\|_0 \le M^{\frac12}\delta\,|\,\|B.\|_0<\delta\,) \le$$
$$P(\,\|w.\|_{\alpha\,,\,c\,\|\xi.^{ij}\|_0^2\,\|B.\|_0^2}\,c\,\|\xi.^{ij}\|_0^{2\alpha}\,\|B.\|_0^{2\alpha} > M\delta^u\,,\,\|\xi.^{ij}\|_0 < M^{\frac12}\delta\,|\,\|B.\|_0<\delta\,)$$
$$\le P(\,\|w.\|_\alpha > c\,M^{\frac12}\delta^{u-2})\cdot P(\,\|B.\|_0<\delta\,)^{-1} \le \exp\left(-\frac{c_\alpha\,M}{\delta^{2(2-u)}}+\frac{c}{\delta^2}\right)\,.$$

Again

(g) $$J_2 \in \mathcal{M}^{\alpha\,,\,0}_u\,.$$

Finally we have to study the martingale part of J_4, the bounded variation being obviously controlled. We can write as above,

$$\int_0^t \alpha_k(x_s)\,B^k_s\,B^i_s\,dB^j_s = w(a(t))\ ,\ a(t) = \int_0^t \alpha_k^2(x_s)\,(B^k_s\,B^i_s)^2\,ds\,.$$

Obviously,

$$P(\,\|\int_0^\cdot \alpha_k(x_s)\,B^k_s\,B^i_s\,dB^j_s\|_\alpha > M\delta^u\,|\,\|B.\|_0<\delta\,) \le$$

$$P(\,\|w.\|_\alpha > c\,M\delta^{u-2})\cdot P(\,\|B.\|_0 < \delta\,)^{-1} \le \exp\,(-\frac{c_\alpha\,M^2}{\delta^{2(2-u)}} + \frac{c}{\delta^2})\,.$$

So,

$$\text{(h)}\qquad\qquad J_4 \in\, \mathcal{M}_u^{\alpha\,,0}\,.$$

Using formulas (a)-(h) we can conclude that $\int_0^\cdot f(x_s)\,d\xi_s^{ij} \in\, \mathcal{M}_u^{\alpha\,,0}$.

(v) We use the same idea, namely we shall apply Ito's formula several times. Firstly, denoting $\frac{\partial f}{\partial x^l} = f_l$,

$$\int_0^t f(x_s)\,dB_s^i = f(x)\,B_t^i + \int_0^t dB_s^i \int_0^s (L_u f)(x_u)\,du +$$

$$\int_0^t dB_s^i \int_0^s f_l(x_u) X_j^l(x_u)\,dB_u^j = S_1 + S_2 + S_3\,.$$

But $\|S_1\|_\alpha \le c\,\|B.\|_\alpha$ and

$$S_2 = B_t^i \int_0^t (L_s f)(x_s)\,ds - \int_0^t B_s^i\,(L_s f)(x_s)\,ds = S_{21} + S_{22}\,,$$

where $\|S_{21}\|_\alpha \le c\,\|B.\|_\alpha$ and $\|S_{22}\|_\alpha \le c\,\|B.\|_0$.

Clearly,

$$S_1\,,\ S_{21}\,,\ S_{22} \in \mathcal{M}_u^{\alpha\,,0}\,.$$

Then, with the same notation as in (iv),

$$S_3 = -B_t^i \int_0^t \alpha_j(x_s)\,dB_s^j + \int_0^t B_s^i\,\alpha_j(x_s)\,dB_s^j + \int_0^t \alpha_j(x_s)\,ds =$$

$$S_{31} + S_{32} + S_{33}\,.$$

By (iv), it is clear that

$$S_{32} = \int_0^t \alpha_j(x_s)\,d\xi_s^{ij} \in\, \mathcal{M}_u^{\alpha\,,0}\,,\ \text{if } i \ne j\,.$$

For $i = j$ we get a term with the same form as S_{33}, terms which are bounded in Hölder norm by a constant. To prove (v), it is sufficient to prove that $S_{31} \in\, \mathcal{M}_u^{\alpha\,,0}$. Note that

$$S_{31} = -B_t^i\,B_t^j\,\alpha_j(x) - B_t^i \int_0^t dB_s^j \int_0^s (L_u\alpha_j)(x_u)\,du -$$

$$B_t^i \int_0^t dB_s^j \int_0^s \alpha_{j,l}(x_u) X_k^l(x_u)\, dB_u^k = S_{311} + S_{312} + S_{313}\,.$$

But $\|S_{311}\|_\alpha \le c\,\|B.\|_0\,\|B.\|_\alpha$ and

$$S_{312} = B_t^i\, B_t^j \int_0^t (L_s\alpha_j)(x_s)\, ds - B_t^i \int_0^t B_s^j\,(L_s\alpha_j)(x_s)\, ds = S_{3121} + S_{3122}\,,$$

where $\|S_{3121}\|_\alpha \le c\,\|B.\|_\alpha\,\|B.\|_0$ and $\|S_{3122}\|_\alpha \le c\,\|B.\|_\alpha\,\|B.\|_0$.

Again

$$S_{311}\,,\ S_{3121}\,,\ S_{3122} \in \mathcal{M}_u^{\alpha\,,0}\,.$$

We denote $\beta_k(x) = -\alpha_{j,l}(x) X_k^l(x)$. Then,

$$S_{313} = B_t^i\, B_t^j \int_0^t \beta_k(x_s)\, dB_s^k - B_t^i \int_0^t B_s^j\, \beta_k(x_s)\, dB_s^k - B_t^i \int_0^t \beta_k(x_s)\, ds =$$

$$S_{3131} + S_{3132} + S_{3133}\,.$$

Arguing as for $S_{32}\,,\ S_{33}$ we see that $S_{3132} = -B_t^i \int_0^t \beta_k(x_s)\, d\xi_s^{jk}\,,\ j \ne k$ and S_{3133} are in $\mathcal{M}_u^{\alpha\,,0}$. We repeat with S_{3131} the computations which we already performed for S_{31} and we see that (with clear notations)

$$S_{31311}\,,\ S_{313121}\,,\ S_{313122}\,,\ S_{313133} \in \mathcal{M}_u^{\alpha\,,0}\,.$$

Then $S_{313132} = B_t^i B_t^j \int_0^t \gamma_l(x_s)\, d\xi_s^{kl}$, $l \ne k$, where $\gamma_l = \beta_m(x) X_l^m(x)$, so S_{313132} satisfies (v) as above.

To control the Hölder norm of S_{313131} we can write

$$S_{313131} = B_t^i B_t^j B_t^k \int_0^t \gamma_l(x_s)\, dB_s^l = B_t^i B_t^j B_t^k\, w(a(t))\ ,\ a(t) = \int_0^t \gamma_l^2(x_s)\, ds\,,$$

where w is a one-dimensional Brownian motion. So,

$$P(\,\|S_{313131}\|_\alpha > M\delta^u \,|\, \|B.\|_0 < \delta\,) \le P(\,\|B.\|_\alpha > M^{\frac12}\delta^{u-\frac12} \,|\, \|B.\|_0 < \delta\,) +$$

$$P(\,\|w.\|_\alpha\, c\,\|B.\|_\alpha\,\|B.\|_0^2 > M\delta^u\,,\ \|B.\|_\alpha \le M^{\frac12}\delta^{u-\frac12} \,|\, \|B.\|_0 < \delta\,) \le$$

$$P(\,\|B.\|_\alpha > M^{\frac12}\delta^{u-\frac12} \,|\, \|B.\|_0 < \delta\,) + \exp\left(-\frac{c_\alpha\, M}{\delta^3} + \frac{c}{\delta^2}\right)\,.$$

From this we can conclude that S_{313131} satisfies (v).

The proof of the lemma is complete.

q.e.d.

4. Support theorem in Hölder norm

Now we are able to extend the support theorem of Stroock-Varadhan for α-Hölder topology. Let us denote by Φ_x the mapping which associates to $h \in L^2 = L^2([0,1], \mathbb{R}^m)$ the solution of the differential equation

$$dy_t = \sum_{j=1}^{m} X_j(t, y_t)\, h_t^j\, dt + X_0(t, y_t)\, dt\,,\; y_0 = x\,. \tag{4.1}$$

(4.2) **Theorem.** *Let* $\alpha \in [0, \frac{1}{2}[$. *For the* $\|\cdot\|_\alpha$*-topology, the support of the probability* P_x *coincide with the closure of* $\Phi_x(L^2)$*, that is,*

$$\mathrm{supp}_\alpha(P_x) = \overline{\Phi_x(L^2)}^{\alpha}\,. \tag{4.3}$$

Proof. To begin with, we note that, for every $\varepsilon > 0$ and $\delta = (\dfrac{\varepsilon}{2^n})^{\frac{1}{u}}$, $u \in]0, 1-2\alpha[$, $n > 0$ integer,

$$P\left(\left\|\int_0^{\cdot} X_k(s, x_s) \circ dB_s^k\right\|_\alpha > \varepsilon \,|\, \|B.\|_0 < \delta\right) =$$

$$P\left(\left\|\int_0^{\cdot} X_k(s, x_s) \circ dB_s^k\right\|_\alpha > 2^n \delta^u \,|\, \|B.\|_0 < \delta\right) \leq$$

$$\sup_{0<\eta\leq 1} P\left(\left\|\int_0^{\cdot} X_k(s, x_s) \circ dB_s^k\right\|_\alpha > 2^n \eta^u \,|\, \|B.\|_0 < \eta\right)\,.$$

Letting $n \uparrow \infty$, by (v) of the Lemma (3.4), we obtain, for every $\varepsilon > 0$,

$$\lim_{\delta\downarrow 0} P\left(\left\|\int_0^{\cdot} X_k(s, x_s) \circ dB_s^k\right\|_\alpha > \varepsilon \,|\, \|B.\|_0 < \delta\right) = 0\,. \tag{4.4}$$

Then we prove that, for every $\varepsilon > 0$,

$$\lim_{\delta\downarrow 0} P(\, \|x. - \Phi_x(0)\|_\alpha < \varepsilon \,|\, \|B.\|_0 < \delta) = 1\,, \tag{4.5}$$

using (4.4) and the following variant of Gronwall's lemma:

(4.6) **Lemma.** *For m and l two functions, put*

$$z_t = z + m(t) + \int_0^t l(z_s)\, ds \,,\; \tilde{z}_t = z + \int_0^t l(\tilde{z}_s)\, ds \,.$$

Suppose that $\|m\|_\alpha \leq \eta$, $m(0) = 0$ and that l is a Lipschitz continuous function with constant L. Then

$$\|z - \tilde{z}\|_\alpha \leq (1 + L)\, e^L\, \eta \,.$$

Proof. By Gronwall's lemma we can immediately write

$$\|z - \tilde{z}\|_0 \leq \eta\, e^L \,.$$

Then,

$$\|z - \tilde{z}\|_{\alpha\,,\,t} \leq \eta + \| \int_0^{\cdot} (l(z_u) - l(\tilde{z}_u))\, du\|_{\alpha\,,\,t} \leq$$

$$\eta + \max_{0 \leq p < q \leq t} \frac{L}{|p - q|^\alpha} | \int_q^p |z_u - \tilde{z}_u|\, du\, | \leq$$

$$\eta + \max_{0 \leq p < q \leq t} \frac{L}{|p - q|^\alpha} \left| \int_q^p (\, |z_q - \tilde{z}_q| + |u - q|^\alpha \|z - \tilde{z}\|_{\alpha\,,\,u}\,)\, du \right| \leq$$

$$\eta + L\, \|z - \tilde{z}\|_0 + L \int_0^t \|z - \tilde{z}\|_{\alpha\,,\,u}\, du \,.$$

Gronwall's lemma ends up the proof of the Lemma (4.6).

q.e.d.

We apply this with $z = \Phi_x(B.)$, $\tilde{z} = \Phi_x(0)$, $m(t) = \int_0^t X_k(s, x_s) \circ dB_s^k$ and $l(x_s) = X_0(s, x_s)$. So, there exists a positive constant K, such that

$$\|\Phi_x(B.) - \Phi_x(0)\,\|_\alpha < K\, \varepsilon \,,$$

provided

$$\left\| \int_0^{\cdot} X_k(s, x_s) \circ dB_s^k \right\|_\alpha \leq \varepsilon \,.$$

Thus we obtain

$$P(\, \|x. - \Phi_x(0)\|_\alpha > \varepsilon \,|\, \|B.\|_0 < \delta\,) =$$

$$P\Big((\,\|x. - \Phi_x(0)\|_\alpha > \varepsilon\,) \cap \Big(\| \int_0^{\cdot} X_k(s,x_s) \circ dB_s^k \|_\alpha > \frac{\varepsilon}{K} \Big) \mid \|B.\|_0 < \delta \Big) \leq$$

$$P\Big(\Big(\| \int_0^{\cdot} X_k(s,x_s) \circ dB_s^k \|_\alpha > \frac{\varepsilon}{K} \Big) \mid \|B.\|_0 < \delta \Big)\,.$$

Now (4.5) is a clear consequence of (4.4).

Finally, Girsanov's formula gives, for any $h \in L^2$ and $\varepsilon > 0$,

$$\lim_{\delta \downarrow 0} P(\, \|\Phi_x(B.) - \Phi_x(h.)\,\|_\alpha < \varepsilon \mid \|B. - h.\|_0 < \delta\,) = 1 \tag{4.7}$$

(as in [S-V], p. 353). But, (4.7) implies

$$P(\, \|\Phi_x(B.) - \Phi_x(h.)\,\|_\alpha < \varepsilon) > 0\,, \text{ for every } \varepsilon > 0\,. \tag{4.8}$$

and, consequently, we obtain the inclusion

$$\mathrm{supp}_\alpha(P_x) \supseteq \overline{\Phi_x(L^2)}^{\,\alpha}\,. \tag{4.9}$$

The converse inclusion is easily obtained using the polygonal approximation of the Brownian motion. For each $n \geq 0$ and $t \geq 0$, we consider

$$t_n = \frac{[2^n]}{2^n}\,,\; t_n^+ = \frac{[2^n]+1}{2^n}\,,\; \dot{B}_t^{(n)} = 2^n (B_{t_n^+} - B_{t_n})\,.$$

Let $(x_t^{(n)})$ be the solution of the equation (4.1) with $\dot{B}_t^{(n)k}$ instead h_t^k. If we denote $P_x^{(n)}$ the law of this solution, it is obvious that

$$x.^{(n)} \in \Phi_x(L^2) \;\text{ and }\; P_x^{(n)}(\,\overline{\Phi_x(L^2)}^{\,\alpha}\,) = 1\,.$$

It suffices to show that P_x is the weak limit of $(P_x^{(n)})$ or, that $(P_x^{(n)})$ is relatively weakly compact with respect to $\|\cdot\|_\alpha$-topology. By classical estimates, for every $p \geq 0$, there exists a positive constant c_p, such that, for every positive integer n and for every $s, t \in [\,0\,,\,1\,]$,

$$E|x_t^{(n)} - x_s^{(n)}|^{2p} \leq c_p |t-s|^p$$

(see for instance [Bi], p. 40). It is easy to see that

$$\sup_n E(\,\|x.^{(n)}\|_{\alpha'}^{2p}\,) \,<\, c\,,\;\text{ if } \alpha' < \frac{p-1}{2p}\,.$$

If we choose p large enough so that $\alpha < \frac{p-1}{2p}$, and if $\alpha' \in]\alpha, \frac{p-1}{2p}[$, it is then clear that the set $K(c) = \{\, z : \|z\|_{\alpha'} < c \,\}$ is compact in $\|\cdot\|_\alpha$-topology, and that, for every $\varepsilon > 0$, there exists a positive constant c_ε, such that,

$$\sup_n P_x^{(n)}(\, K(c_\varepsilon)\,) < \varepsilon\,.$$

So, $(P_x^{(n)})$ is tight.

The proof of the Theorem (4.2) is complete.

q.e.d.

Appendix

We give now another proof of a variant of (1.10) (or (1.19)), when $\beta = 0$, which does not require the use of Ciesielski's theorem (that is (1.4) and (1.5)) nor the correlation inequality.

(A.1) **Theorem.** *Let (r, R) be a couple of real positive numbers. For every $a' < a$ and $b' > b$, there exists a constant c, such that, if $\frac{R^{a'}}{r^{b'}} > c$, then*

$$P\,(\,(\,\|w\|_\alpha > R\,)\,\cap\,(\,\|w\|_\beta < r\,)\,) \leq \exp\left(-\frac{1}{2}\,\frac{R^{\frac{1-2\beta}{\alpha-\beta}}}{r^{\frac{1-2\alpha}{\alpha-\beta}}}\right)\,, \tag{A.2}$$

for $0 \leq \beta < \alpha < \frac{1}{2}$.

Proof. Put

$$\eta = \left(\frac{r}{R}\right)^{\frac{1}{\alpha-\beta}}\,.$$

Then, if $\|w\|_\beta < r$,

$$\sup_{s<t,t-s>\eta} \frac{|w_t - w_s|}{|t-s|^\alpha} \leq R\,.$$

Thus we obtain

$$\begin{gathered}(\,(\,\|w\|_\alpha > R\,)\,\cap\,(\,\|w\|_\beta < r\,)\,) \subset \\ \left(\left(\sup_{s<t\leq s+\eta} \frac{|w_t - w_s|}{|t-s|^\alpha} \geq R\right) \cap (\,\sup_t |w_t| < r\,)\right) \subset \\ \left(\sup_{s<t\leq s+\eta} \frac{|w_t - w_s|}{|t-s|^\alpha} \geq R\right) = (\,\sup_{v\in D} |X_v^\alpha| \geq R\,)\,.\end{gathered}$$

Here $v=(s,t)$, $D=\{v: s<t\leq s+\eta\}$ and $X_v^\alpha=\frac{w_t-w_s}{|t-s|^\alpha}$ is a two-parameter gaussian variable.

Now, we can estimate

$$P\,(\,(\,\|w\|_\alpha > R\,)\,\cap\,(\,\|w\|_\beta < r\,)\,)\leq$$

$$P(\sup_{v\in D}|X_v^\alpha|\geq R\,)\leq \exp\left(-\frac{(R-M_\alpha)^2}{2X_\alpha^2}\right)\,,$$

where the last inequality is valid when $R\geq M_\alpha$ (see [L-T], p. 57). Here

$$0<M_\alpha=E(\sup_{v\in D}|X_v^\alpha|\,)\leq E(\,\|w\|_\alpha\,)<\infty$$

and

$$X_\alpha^2=\sup_{v\in D}E(\,(X_v^\alpha)^2\,)=\eta^{1-2\alpha}\,.$$

So, we get

$$P\,(\,(\,\|w\|_\alpha > R\,)\,\cap\,(\,\|w\|_\beta < r\,)\,)\leq$$

$$\exp\left(-\frac{R^2}{2\eta^{1-2\alpha}}\right)=\exp\left(-\frac{1}{2}\,\frac{R^{\frac{1-2\beta}{\alpha-\beta}}}{r^{\frac{1-2\alpha}{\alpha-\beta}}}\right)\,,\ \beta\geq 0\,.$$

The restriction $R\geq M_\alpha$ may be weakened as follows. Take $\alpha'>\alpha$ and write

$$\left(\sup_{s<t\leq s+\eta}\frac{|w_t-w_s|}{|t-s|^\alpha}\geq R\right)=\left(\sup_{s<t\leq s+\eta}\frac{|w_t-w_s|}{|t-s|^{\alpha'}}\cdot|t-s|^{\alpha'-\alpha}\geq R\right)\subset$$

$$\left(\sup_{s<t\leq s+\eta}\frac{|w_t-w_s|}{|t-s|^{\alpha'}}\geq R\,\eta^{\alpha-\alpha'}\right)=\left(\sup_{s<t\leq s+\eta}\frac{|w_t-w_s|}{|t-s|^{\alpha'}}\geq\frac{R^{\frac{\alpha'-\beta}{\alpha-\beta}}}{r^{\frac{\alpha'-\alpha}{\alpha-\beta}}}\right)\,.$$

Now, we need only

$$\frac{R^{\frac{\alpha'-\beta}{\alpha-\beta}}}{r^{\frac{\alpha'-\alpha}{\alpha-\beta}}}>E(\,\|w\|_{\alpha'}\,)=M_{\alpha'}\,,$$

and the proof of the theorem is complete.

q.e.d.

Clearly, the Theorem (A.1) implies that

$$P(\,\|w.\|_\alpha > R \,|\, \|w.\|_0 < r\,) = \frac{P\,(\,(\,\|w\|_\alpha > R\,) \bigcap (\,\|w\|_0 < r\,))}{P(\,\|w\|_0 < r\,)} \leq$$

$$\exp\left(-\frac{1}{2}\,\frac{R^{\frac{1}{\alpha}}}{r^{\frac{1}{\alpha}-2}} \right) \exp\left(\frac{\pi^2}{8} \cdot \frac{1}{r^2} \right) .$$

If r is small we need the condition $\alpha < \frac{1}{4}$ for an interesting estimate.

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