Supersymmetry for probabilists

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Abstract

These are notes for a short introduction to supersymmetry from a probabilistic perspective, given at the Fields Institute in Toronto in 2018.

Contents

1 Introduction

Many problems in statistical mechanics can be reduced to understanding the asymptotics of measures

$$\frac{1}{Z_N} e^{-H_N(\varphi)} \prod_{i=1}^{N} d\varphi_i. \tag{1.1}$$

These include in particular spin systems such as the Ising and $O(n)$ models, models of interacting particles describing states from solids through gases, and many more. Supersymmetry extends this idea in a direction that is relevant for the description of disordered systems such as random operators and interacting random walks. In the version most relevant for statistical mechanics, it roughly corresponds to replacing measures by differential forms. Models for which supersymmetry is particularly relevant include:

- Random walks (simple random walks, self-avoiding walks, edge- and vertex-reinforced walks);
- Random matrices and quantum chaos;
- Stochastic dynamics.

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For much of this, only the surface is understood mathematically. These lectures provide a concrete introduction to some of the basic principles of supersymmetry from the point of view of probability theory, focusing on the aspect of random walks. For the mathematics of superanalysis, a thorough reference is [3]. In the physics literature, excellent treatments exist focusing on random matrix theory [8, 13]. For mathematical progress in this direction, see [12] and [11] and references given there. The supersymmetries discussed in this lecture are *internal supersymmetries*; these are much more approachable than spatial supersymmetries which are relevant in particle physics.

Acknowledgements. These notes are mainly based on [1, Chapter 11] and [2].

2 Supersymmetry

2.1 Integration of differential forms

We begin with the important example of Euclidean space $\mathbb{R}^N$ with coordinates $x_1, \dots, x_N$. The coordinates can be viewed as functions $x_i : \mathbb{R}^N \to \mathbb{R}$ in the algebra $C^\infty(\mathbb{R}^N)$. A differential form on $\mathbb{R}^N$ can be written as

$$F = F_0 + \cdots + F_N \tag{2.1}$$

where $F_0 \in C^\infty(\mathbb{R}^N)$ is a 0-form, i.e., an ordinary function, and F_p is a p-form, i.e., a sum of forms

$$f(x)\, dx_{i_1} \wedge \cdots \wedge dx_{i_p} \tag{2.2}$$

where $f \in C^\infty(\mathbb{R}^N)$ and the differentials dx_i are the generators of a Grassmann algebra (also called exterior algebra). This means that they are multiplied with the anti-commuting wedge product:

$$dx_i \wedge dx_j = -dx_j \wedge dx_i. \tag{2.3}$$

In particular, $dx_i \wedge dx_i = 0$. Later, the $\wedge$ will often by omitted.

The form F_p is the degree-p part of F and a form F has degree p if $F = F_p$. A differential form is even if it is a sum of p-forms with all p even and it is odd if it sum of p-form will all p odd. By antisymmetry, there are no forms of degree greater than N. Thus a degree-N form is said to be of top-degree and it can be written as

$$F(x) = f(x)\, dx_1 \wedge \cdots \wedge dx_N. \tag{2.4}$$

The order of the differentials determines an overall sign. The integral of an N-form is defined by

$$\int_{\mathbb{R}^N} F = \int_{\mathbb{R}^N} f(x)\, dx_1 \cdots dx_N \tag{2.5}$$

where the right-hand side is a usual Lebesgue integral. For a p-form with $p < N$, we set $\int F = 0$ and extend the definition of the integral linearly to the space of differential forms.

Change of variables. The differential notation and the use of the wedge product is consistent and motivated by the following change of variable formula. Let $\Phi : \mathbb{R}^N \to \mathbb{R}^N$ be a diffeomorphism. Then

$$\int f(x_1, \dots, x_N)\, dx_1 \wedge \cdots \wedge dx_N = \int f(\Phi_1(x), \dots, \Phi_N(x))(\det D\Phi)\, dx_1 \wedge \cdots \wedge dx_N \tag{2.6}$$

$$= \int f(\Phi_1(x), \dots, \Phi_N(x))\, d\Phi_1(x) \wedge \cdots \wedge d\Phi_N(x) \tag{2.7}$$

where

$$d\Phi_i(x) = \sum_j \frac{\partial \Phi(x)}{\partial x_j} dx_j. \tag{2.8}$$

Complex coordinates. It can be useful to identify $\mathbb{R}^2$ with $\mathbb{C}$. Denote the coordinates of $\mathbb{R}^2$ by x, y with differentials dx and dy. To identify $\mathbb{R}^2$ with $\mathbb{C}$ set

$$z = x + iy, \quad \bar{z} = x - iy, \quad dz = dx + idy, \quad d\bar{z} = dx - idy. \tag{2.9}$$

Since $dx \wedge dx = dy \wedge dy = 0$ then

$$d\bar{z} \wedge dz = 2i(dx \wedge dy), \tag{2.10}$$

and the standard integral can be written as

$$\int f(x,y)\, dx \wedge dy = \int f(\operatorname{Re} z, \operatorname{Im} z) \frac{d\bar{z} \wedge dz}{2i}. \tag{2.11}$$

2.2 Berezin integral

Let Λ^M be a Grassmann algebra with generators $\xi_1, \dots, \xi_M$. Thus Λ^M is a unital associate algebra generated by $(\xi_i)_i$ which satisfy the anticommutation relations

$$\xi_i \xi_j + \xi_j \xi_i = 0. \tag{2.12}$$

Let $\Lambda^M(\mathbb{R}^N)$ be the algebra of smooth functions from $\mathbb{R}^N$ into Λ^M.

Example 2.1. *The differentials $\xi = dx_i$ are an instance of a Grassmann algebra and the algebra of differential forms on $\mathbb{R}^N$ is then $\Lambda^N(\mathbb{R}^N)$.*

We use the term *form* for elements of $\Lambda^M(\mathbb{R}^N)$ also if $N \neq M$. The notations of degree defined for differential forms extend to this more general context.

Exercise 2.2. *Find $2^M \times 2^M$ matrices $\xi_1, \dots, \xi_M$ that generate a version of Λ^M as a matrix algebra. Hint: the Clifford–Jordan–Wigner representation of the Grassmann algebra is*

$$\xi_i = \bigotimes_{j=1}^{i-1} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \otimes \bigotimes_{j=i+1}^{m} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{2.13}$$

The left-derivative $\partial_{\xi_i} = \frac{\partial}{\partial \xi_i} : \Lambda^M \to \Lambda^M$ is the linear map determined by

$$\frac{\partial}{\partial \xi_i}(\xi_i F) = F \quad \text{if } \xi_i F \neq 0, \qquad \frac{\partial}{\partial \xi_i} 1 = 0. \tag{2.14}$$

Note that ∂_{ξ_i} is an anti-derivation: if F is a p-form, then

$$\partial_{\xi_i}(FG) = (\partial_{\xi_i} F)G + (-1)^p F(\partial_{\xi_i} G) \tag{2.15}$$

and that it extends naturally to $\Lambda^M(\mathbb{R}^N)$ by acting pointwise.

Example 2.3. *Let $F \in \Lambda^N(\mathbb{R}^N)$ be a differential form and write $\xi_i = dx_i$. Then*

$$\int F = \int_{\mathbb{R}^N} dx_1 \cdots dx_N\, \partial_{\xi_N} \cdots \partial_{\xi_1} F = \int_{\mathbb{R}^N} dx\, \partial_\xi F \tag{2.16}$$

where the left-hand side is the integral as a differential form in the previous sense.

The notation on the right-hand side is called the Berezin integral and also generalizes to $N \neq M$. It is further useful because one can change variables in dx and ∂_ξ separately (see later).

Definition 2.4. *Let $g \in C^\infty(\mathbb{R}^k)$ and $F^1, \dots F^k \in \Lambda^M(\mathbb{R}^N)$ be even. Then $g(F^1, \dots, F^k) \in \Lambda^M(\mathbb{R}^N)$ is defined by*

$$g(F^1, \dots, F^k) = \sum_\alpha \frac{1}{\alpha!} g^{(\alpha)}(F_0^1, \dots, F_0^k)(F - F_0)^\alpha \tag{2.17}$$

where the right-hand is the formal Taylor expansion about the degree-0 part

$$g(F_0^1, \dots, F_0^k) + \sum_{i=1}^k g_i(F_0^1, \dots, F_0^k)(F^i - F_0^i) + \sum_{i,j=1}^k \frac{1}{2} g_{ij}(F_0^1, \dots, F_0^k)(F^i - F_0^i)(F^j - F_0^j) + (\cdots). \tag{2.18}$$

and g_i is the derivative with respect to the i-th argument, and so on.

Note that the formal Taylor expansion is finite since $N < \infty$ and there are not elements in Λ^N of degree greater than N, and that the ordering of the products does not matter since all forms are assumed even.

Example 2.5.

$$e^{-x_1^2 - \xi_1\xi_2} = e^{-x_1^2}(1 - \xi_1\xi_2). \tag{2.19}$$

2.3 Gaussian integrals

Let $A \in \mathbb{R}^{N\times N}$ be positive definite and set $C = A^{-1}$. The Gaussian measure on $\mathbb{R}^N$ with covariance C has density

$$p_C(dx) = e^{-\frac{1}{2}(x,Ax)}(\det A)^{1/2} \prod_{i=1}^N \frac{dx_i}{\sqrt{2\pi}}. \tag{2.20}$$

The normalisation is such that

$$\int p_C(dx) = 1, \quad \int x_i \, p_C(dx) = 0, \quad \int x_i x_j \, p_C(dx) = C_{ij}. \tag{2.21}$$

Two copies of the Gaussian measure give the complex Gaussian measure on $\mathbb{C}^N \cong \mathbb{R}^{2N}$ defined by

$$p_C(d\bar{z}, dz) = e^{-\frac{1}{2}(x,Ax) - \frac{1}{2}(y,Ay)}(\det A) \prod_{i=1}^N \frac{dx_i \, dy_i}{2\pi} = e^{-\frac{1}{2}(z,A\bar{z})}(\det A) \prod_{i=1}^N \frac{d\bar{z}_i \, dz_i}{4\pi i}. \tag{2.22}$$

Its normalisation is here such that

$$\int z_i \bar{z}_j \, p_C(d\bar{z}, dz) = 2C_{ij}. \tag{2.23}$$

Therefore the usual normalisation in the complex case is to replace A by $2A$ and thus C by $C/2$. To compare better with the real case, we will not use these factors of two.

Now fix any branch of the complex square root and write

$$\zeta_i = \frac{dz_i}{\sqrt{2\pi i}}, \quad \bar{\zeta}_i = \frac{d\bar{z}_i}{\sqrt{2\pi i}}. \tag{2.24}$$

Then by definition of the determinant,

$$(\det A) \prod_{i=1}^N \frac{d\bar{z}_i \, dz_i}{4\pi i} = (\det A) \prod_{i=1}^N \frac{\bar{\zeta}_i \zeta_i}{2} = \frac{1}{N!} \left(\sum_{i,j=1}^N A_{ij} \frac{\bar{\zeta}_i \zeta_j}{2} \right)^N = e^{\frac{1}{2}(\bar{\zeta}, A\zeta)}\Big|_{2N} = e^{-\frac{1}{2}(\zeta, A\bar{\zeta})}\Big|_{2N}. \tag{2.25}$$

Thus the complex Gaussian measure can be written as

$$e^{-\frac{1}{2}(z,A\bar{z})-\frac{1}{2}(\zeta,A\bar{\zeta})}\Big|_{2N}. \tag{2.26}$$

In particular, by normalisation of the Gaussian measure, for any choice of A,

$$\int e^{-\frac{1}{2}(z,A\bar{z})-\frac{1}{2}(\zeta,A\bar{\zeta})} = 1. \tag{2.27}$$

This is an instance of the general principle of localisation of supersymmetric integrals.

2.4 Localisation

We now consider the $\Lambda^{2N}(\mathbb{R}^{2N})$ and use complex coordinates. Define

$$\frac{\partial}{\partial z_i} = \frac{1}{2}\left(\frac{\partial}{\partial x_i} - i\frac{\partial}{\partial y_i}\right), \quad \frac{\partial}{\partial \bar{z}_i} = \frac{1}{2}\left(\frac{\partial}{\partial x_i} + i\frac{\partial}{\partial y_i}\right), \tag{2.28}$$

and define ∂_{ζ_i} and $\partial_{\bar{\zeta}_i}$ to be the antiderivations on Λ^{2N} such that

$$\frac{\partial}{\partial \zeta_i}\zeta_j = \frac{\partial}{\partial \bar{\zeta}_i}\bar{\zeta}_j = \delta_{ij}, \qquad \frac{\partial}{\partial \zeta_i}\bar{\zeta}_j = \frac{\partial}{\partial \bar{\zeta}_i}\zeta_j = 0. \tag{2.29}$$

The *supersymmetry generator* $Q : \Lambda^{2N}(\mathbb{R}^{2N}) \to \Lambda^{2N}(\mathbb{R}^{2N})$ is defined by

$$Q = \sum_{i=1}^{N}\left(\zeta_i\frac{\partial}{\partial z_i} + \bar{\zeta}_i\frac{\partial}{\partial \bar{z}_i} - z_i\frac{\partial}{\partial \zeta_i} + \bar{z}_i\frac{\partial}{\partial \bar{\zeta}_i}\right). \tag{2.30}$$

The form $F \in \Lambda^{2N}(\mathbb{R}^{2N})$ is defined to be *supersymmetric* (or Q-closed) if $QF = 0$ and it is Q-exact if $F = QG$ for some form $G \in \Lambda^{2N}(\mathbb{R}^{2N})$.

Example 2.6. *Q formally exchanges the even and odd generators of $\Lambda^{2N}(\mathbb{R}^{2N})$:*

$$Qz_i = \zeta_i, \quad Q\bar{z}_i = \bar{\zeta}_i, \quad Q\zeta_i = -z_i, \quad Q\bar{\zeta}_i = \bar{z}_i. \tag{2.31}$$

Example 2.7. *The forms*

$$\tau_{ij} = \frac{1}{2}(z_i\bar{z}_j + \zeta_i\bar{\zeta}_j + z_j\bar{z}_i + \zeta_j\bar{\zeta}_i) \tag{2.32}$$

and Q-closed and Q-exact with

$$\tau_{ij} = Q\lambda_{ij}, \quad \lambda_{ij} = \frac{1}{2}(z_i\bar{\zeta}_j + z_j\bar{\zeta}_i). \tag{2.33}$$

Much of the magic of supersymmetry is due to the following fundamental *Localisation Theorem.*

Theorem 2.8. *Let the form $F \in \Lambda^{2N}(\mathbb{R}^{2N})$ be supersymmetric and integrable. Then*

$$\int F = F_0(0) \tag{2.34}$$

where the right-hand side is the degree-0 part of F evaluated at 0.

In preparation of the proof, we need the following chain rule for Q.

Lemma 2.9. *The supersymmetry generator Q obeys the chain rule for even forms, in the sense that if $K = (K_j)_{j \le J}$ is a finite collection of even forms, and if $f : \mathbb{R}^J \to \mathbb{C}$ is C^∞, then*

$$Q(f(K)) = \sum_{j=1}^{J} f_j(K) Q K_i, \tag{2.35}$$

where f_i denotes the partial derivative.

Proof. Suppose first that K is a collection of zero forms. Then

$$Qf(K) = \sum_{i=1}^{N} \left[\zeta_i \frac{\partial f(K)}{\partial z_i} + \bar{\zeta}_i \frac{\partial f(K)}{\partial \bar{z}_i} \right] = \sum_{j=1}^{J} f_j(K) \sum_{i=1}^{N} \left[\zeta_i \frac{\partial K_j}{\partial z_i} + \bar{\zeta}_i \frac{\partial K_j}{\partial \bar{z}_i} \right], \tag{2.36}$$

where the second equality follows from the chain rule for zero-forms. The right-hand side is $\sum_j f_j(K) Q K_j$, so this proves (2.35) for 0-forms and we may assume now that K is higher degree.

Let ε_j be the multi-index that has j^{th} component 1 and all other components 0. Let $K^0 = (K_j^0)_{j \in J}$ denote the zero-degree part of K. By the fact that Q is an anti-derivation, and the chain rule applied to zero-forms,

$$\begin{aligned} Qf(K) &= \sum_{\alpha} \frac{1}{\alpha!} [Q f^{(\alpha)}(K^0)](K - K^0)^\alpha + \sum_{\alpha} \frac{1}{\alpha!} f^{(\alpha)}(K^0) Q[(K - K^0)^\alpha] \\ &= \sum_{\alpha} \frac{1}{\alpha!} \sum_{j=1}^{J} f^{(\alpha+\varepsilon_j)}(K^0)[Q K_i^0](K - K^0)^\alpha + \sum_{\alpha} \frac{1}{\alpha!} f^{(\alpha)}(K^0) Q[(K - K^0)^\alpha]. \end{aligned} \tag{2.37}$$

Since Q is an anti-derivation,

$$Q(K - K^0)^\alpha = \sum_{j=1}^{J} \alpha_j (K - K^0)^{\alpha - \varepsilon_j} [Q K_j - Q K_j^0]. \tag{2.38}$$

The first term on the right-hand side of (2.37) is cancelled by the contribution to the second term of (2.37) due to the second term of (2.38). The remaining contribution to the second term of (2.37) due to the first term of (2.38) then gives

$$Qf(K) = \sum_{j=1}^{J} \left(\sum_{\alpha} \frac{1}{\alpha!} f^{(\alpha)}(K^0) \alpha_j (K - K^0)^{\alpha - \varepsilon_j} \right) Q K_j = \sum_{j=1}^{J} f_j(K) Q K_j \tag{2.39}$$

as required. □

Corollary 2.10. *Let $\tau = (\tau_{ij})_{i,j=1}^{N}$ be the collection of forms* (2.32)*. Then for any smooth function $f : \mathbb{R}^{N^2} \to \mathbb{R}$ with sufficient decay,*

$$\int f(\tau) = f(0). \tag{2.40}$$

Proof. Let $F = f(\tau)$. Then $F_0(0) = f(0)$ and $QF = \sum_{ij} f_{ij}(\tau) Q \tau_{ij} = 0$ by the chain rule for Q below. The claim thus follows from the localisation theorem above. □

Proof of Theorem 2.8. Any integrable form K can be written as $K = \sum_\alpha K^\alpha \zeta^\alpha$, where ζ^α is a monomial in the $\zeta_i, \bar{\zeta}_i$ where $i = 1, \dots, N$, and K^α is an integrable function of $z, \bar{z}$. To emphasise this, we write $K = K(z, \bar{z}, \zeta, \bar{\zeta})$.

Step 1. Let $S = \sum_i (z_i \bar{z}_i + \zeta_i \wedge \bar{\zeta}_i)$. We prove the following version of Laplace's Principle:

$$\lim_{t \to \infty} \int e^{-tS} K = K_0(0). \tag{2.41}$$

Let $t > 0$. We make the change of variables $z_i = \frac{1}{\sqrt{t}} z_i'$ and $\zeta_i = \frac{1}{\sqrt{t}} \zeta_i'$; since ζ_i is proportional to dz_i this correctly implements the change of variables. Let $\omega = -\sum_{x \in \Lambda} \zeta_x \wedge \bar{\zeta}_x$. After dropping the primes, we obtain

$$\int e^{-tS} K = \int e^{-\sum_x z_x \bar{z}_i + \omega} K(\tfrac{1}{\sqrt{t}} z, \tfrac{1}{\sqrt{t}} \bar{z}, \tfrac{1}{\sqrt{t}} \zeta, \tfrac{1}{\sqrt{t}} \bar{\zeta}). \tag{2.42}$$

To evaluate the right-hand side, we expand e^{ω} and and obtain

$$\int e^{-tS} K = \sum_{n=0}^{N} \int e^{-\sum_i z_i \bar{z}_i} \frac{1}{n!} \omega^n K(\tfrac{1}{\sqrt{t}} z, \tfrac{1}{\sqrt{t}} \bar{z}, \tfrac{1}{\sqrt{t}} \zeta, \tfrac{1}{\sqrt{t}} \bar{\zeta}). \tag{2.43}$$

We write $K = K^0 + G$, where $G = K - K^0$ contains no degree-zero part. The contribution of K^0 to to (2.43) involves only the $n = N$ term and equals

$$\int e^{-tS} K^0 = \int e^{-\sum_i z_i \bar{z}_i} \frac{1}{N!} \omega^N K^0(\tfrac{1}{\sqrt{t}} z, \tfrac{1}{\sqrt{t}} \bar{z}), \tag{2.44}$$

so by the continuity of K_0,

$$\lim_{t \to \infty} \int e^{-tS} K_0 = K_0(0) \int e^{-\sum_i z_i \bar{z}_i} \frac{1}{N!} \omega^N = K_0(0) \int e^{-S}. \tag{2.45}$$

By (2.27) with $A = \mathrm{id}$, this proves that

$$\lim_{t \to \infty} \int e^{-tS} K_0 = K_0(0). \tag{2.46}$$

To complete the proof of (2.41), it remains to show that $\lim_{t \to \infty} \int e^{-tS} G = 0$. As above,

$$\int e^{-tS} G = \sum_{n=0}^{N} \int e^{-\sum_i z_i \bar{z}_i} \frac{1}{n!} \omega^n \, G\left(\tfrac{1}{\sqrt{t}} z, \tfrac{1}{\sqrt{t}} \bar{z}, \tfrac{1}{\sqrt{t}} \zeta, \tfrac{1}{\sqrt{t}} \bar{\zeta}\right). \tag{2.47}$$

Since G has no degree-zero part, the term with $n = N$ is zero. Terms with smaller n require factors $\zeta \bar{\zeta}$ from G, which carry inverse powers of t. They therefore vanish in the limit, and the proof of (2.41) is complete.

Step 2. The Laplace approximation is exact:

$$\int e^{-tS} K \quad \text{is independent of } t \geq 0. \tag{2.48}$$

To prove this, recall that $\tau_i = Q\lambda_i$. Let $\lambda = \sum_i \lambda_i$. Then

$$S = \sum_i \tau_i = \sum_i Q\lambda_i = Q\lambda. \tag{2.49}$$

Also, $Qe^{-S} = 0$ by Example **??**, and $QK = 0$ by assumption. Therefore,

$$\frac{d}{dt} \int e^{-tS} K = -\int e^{-tS} S K = -\int e^{-tS} (Q\lambda) K = -\int Q\left(e^{-tS} \lambda K\right) = 0, \tag{2.50}$$

since the integral of any Q-exact form is zero.

Step 3. Finally, we combine Laplace's Principle (2.41) and the exactness of the Laplace approximation (2.48), to obtain the desired result

$$\int K = \lim_{t \to \infty} \int e^{-tS} K = K_0(0). \tag{2.51}$$

This completes the proof. □

2.5 Complex coordinates and change of generators

2.5.1. Complex coordinates. The above can also be expressed in real instead of complex coordinates. Let

$$\xi_i = \frac{dx_i}{\sqrt{2\pi}}, \quad \eta_i = \frac{dy_i}{\sqrt{2\pi}}. \tag{2.52}$$

Then

$$\zeta = \frac{1}{\sqrt{i}}(\xi + i\eta), \qquad \frac{\partial}{\partial \zeta} = \frac{\sqrt{i}}{2}\left(\frac{\partial}{\partial \xi} - i\frac{\partial}{\partial \eta}\right), \tag{2.53}$$

$$\bar\zeta = \frac{1}{\sqrt{i}}(\xi - i\eta), \qquad \frac{\partial}{\partial \bar\zeta} = \frac{\sqrt{i}}{2}\left(\frac{\partial}{\partial \xi} + i\frac{\partial}{\partial \eta}\right). \tag{2.54}$$

In particular, $\zeta\bar\zeta = 2\eta\xi$, $\partial_\zeta \partial_{\bar\zeta} = \frac{1}{2}\partial_\eta\partial_\xi$ and

$$\begin{aligned}(z, A\bar z) + (\zeta, A\bar\zeta) &= \sum_{i,j} A_{ij}(x_i x_j + y_i y_j - \xi_i \eta_j + \eta_i \xi_j)\\ &= \sum_{i,j} A_{ij} \begin{pmatrix} x \\ y \\ \xi \\ \eta \end{pmatrix}^T \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right) \begin{pmatrix} x \\ y \\ \xi \\ \eta \end{pmatrix}. \end{aligned} \tag{2.55}$$

The supersymmetry generator can be written as

$$Q = \frac{1}{\sqrt{i}} \sum_i \left(\xi_i \frac{\partial}{\partial x_i} + \eta_i \frac{\partial}{\partial y_i} - x_i \frac{\partial}{\partial \eta_i} + y_i \frac{\partial}{\partial \xi_i}\right). \tag{2.56}$$

2.5.2. Change of generators. In terms of the coordinate maps $x^\alpha : \mathbb{R}^N \to \mathbb{R}$, every $f \in C^\infty(\mathbb{R}^N)$ can be written as

$$f = f(x_1, \dots, x_N). \tag{2.57}$$

In general, forms $F \in \Lambda^M(\mathbb{R}^N)$ can be written in the form

$$F = f_\varnothing(x^1, \dots, x^N) + f_1(x^1, \dots, x^N)\xi^1 + \cdots \tag{2.58}$$

and we sometimes write $F(x^1, \dots, x^N, \xi^1, \dots, \xi^M)$ to denote that F is a form that can be written in this way. Note however that the ξ^i do not commute the order of their products matters.

Definition 2.11. *Collections of even elements $(x^\alpha)_\alpha$ and odd elements $(\xi^\alpha)_\alpha$ are a set of generators for $\Lambda^M(\mathbb{R}^N)$ if every $F \in \Lambda^M(\mathbb{R}^N)$ can be written in the form above.*

It is possible to change between sets of generators. This is an extension of the change of variable formula. Recall the following example for change of variables in the context of differential forms.

Example 2.12. *Let $\Phi : \mathbb{R}^N \to \mathbb{R}^N$ be a diffeomorphism. Then*

$$y_i = \Phi_i(x^1, \dots, x^N), \quad \eta_i = dy_i = \sum_j \frac{\partial \Phi_i}{\partial x_j}(x^1, \dots, x^N)\, dx_j \tag{2.59}$$

are generators for $\Lambda^N(\mathbb{R}^N)$ and the change of variables formula becomes

$$\int dx\, \partial_\xi\, F(x, \xi) = \int F(x, dx) = \int F(y, dy) = \int dy\, \partial_\eta\, F(y, \eta). \tag{2.60}$$

There is a general version of the change of variables formula to change generators of $\Lambda^M(\mathbb{R}^N)$ that does not require that the even and odd generators change together; see [3].

Example 2.13. *Let $\Phi : \mathbb{R}^N \to \mathbb{R}^N$ be a diffeomorphism. Then the following is a set of generators:*

$$y_i = \Phi_i(x^1, \dots, x^N), \quad \eta_i = \xi_i \tag{2.61}$$

and the usual change of variable formula gives

$$\int dx\, \partial_\xi\, F(x,\xi) = \int dy\, \partial_\eta\, (\det D\Phi)\, F(y,\eta). \tag{2.62}$$

Example 2.14. *Let x, ξ_1, ξ_2 be generators for $\Lambda^2(\mathbb{R})$. Then*

$$x + g(x)\xi_1\xi_2, \quad \xi_1, \quad \xi_2 \tag{2.63}$$

is also a set of generators and

$$\int dx\, \partial_{\xi_1}\partial_{\xi_2}\, F(x,\xi_1,\xi_2) = \int dx\, \partial_{\xi_1}\partial_{\xi_2}\, F(x + g(x)\xi_1\xi_2, \xi_1, \xi_2)(1 + g'(x)\xi_1\xi_2). \tag{2.64}$$

Proof. By definition,

$$F(x + g(x)\xi_1\xi_2, \xi_1, \xi_2) = F(x,\xi_1,\xi_2) + F'(x,\xi_1,\xi_2)g(x)\xi_1\xi_2. \tag{2.65}$$

By integration by parts, therefore

$$\int dx\, \partial_{\xi_1}\partial_{\xi_2}\, F(x + g(x)\xi_1\xi_2, \xi_1, \xi_2) = \int dx\, \partial_{\xi_1}\partial_{\xi_2}\, F(x,\xi_1,\xi_2)(1 - g'(x)\xi_1\xi_2). \tag{2.66}$$

Since $F(x + g(x)\xi_1\xi_2, \xi_1, \xi_2)g'(x)\xi_1\xi_2 = F(x,\xi_1,\xi_2)g'(x)\xi_1\xi_2$ the claim follows. □

3 Random walk and supersymmetry

3.1 Laplacian and simple random walk

From now on, we will often write Λ for $\{1,\dots,N\}$ and $\mathbb{R}^N \cong \mathbb{R}^\Lambda$ to emphasize that the points in Λ should be regarded to be the vertices of a graph.

Let $(\beta_{ij})_{i,j\in\Lambda}$ be non-negative symmetric edge weights. For $f \in \mathbb{R}^\Lambda$, define the *Laplacian* by

$$(\Delta_\beta f)_i = \sum_j \beta_{ij}(f_j - f_i). \tag{3.1}$$

Thus

$$(f, -\Delta_\beta f) = \sum_i f_i(-\Delta_\beta f)_i = \frac{1}{2}\sum_{i,j} \beta_{ij}(f_i - f_j)^2. \tag{3.2}$$

3.2 Gaussian free field

The operator Δ_β is the generator of a continuous-time simple random walk on a finite state space. This is a walk $(X_t)_{t\geqslant 0}$ with jump rates

$$\mathbb{P}(X_{t+\delta t} = j | X_t = i) = \beta_{ij}\delta t + o(\delta t). \tag{3.3}$$

The local time at vertex j of the simple random walk is the time spent at that vertex:

$$L_t^j = \int_0^t 1_{X_s = j}\, ds. \tag{3.4}$$

We also also associate a random field to β. For $\varphi \in \mathbb{R}^{nN}$ with $\varphi_i = (x_i^1, \dots, x_i^n)$,

$$(\varphi, -\Delta_\beta \varphi) = \sum_{\alpha=1}^n (\varphi^\alpha, -\Delta_\beta \varphi^\alpha). \tag{3.5}$$

The n-component *Gaussian Free Field* (GFF) associated to the edge weights β with mass $m > 0$ is the probability measure on $\mathbb{R}^{nN}$ given by

$$\frac{1}{Z} e^{-\frac{1}{2}(\varphi, -\Delta_\beta\varphi) - \frac{1}{2}m^2(\varphi,\varphi)}\, d\varphi. \tag{3.6}$$

We write the components of φ as

$$\varphi = (x^1, \dots, x^n), \quad d\varphi = dx_1^1 \dots dx_N^n. \tag{3.7}$$

There is an intimate connection between the GFF and the SRW.

Theorem 3.1 (BFS [4], Dynkin [7])**.**

$$\int d\varphi\, e^{-\frac{1}{2}(\varphi,-\Delta\varphi)} g(\frac{1}{2}|\varphi|^2) x_i^1 x_j^1 = \int d\varphi\, e^{-\frac{1}{2}(\varphi,-\Delta\varphi)} \int_0^\infty dt\, \mathbb{E}_i(1_{X_t=j}\, g(\frac{1}{2}|\varphi|^2 + L_t)) \tag{3.8}$$

The theorem follows from the following observation and lemma. The joint process (X_t, L_t) is again a Markov process with state space $\Lambda \times \mathbb{R}^\Lambda$ and generator

$$\mathcal{L}g(i,\ell) = \Delta_\beta g(i,\ell) + \frac{\partial}{\partial \ell_i} g(i,\ell), \tag{3.9}$$

where Δ_β acts pointwise in the ℓ-coordinate.

Lemma 3.2. *For every smooth $g : \Lambda \times \mathbb{R}^\Lambda \to \mathbb{R}$ with sufficient decay,*

$$-\sum_j \int d\varphi\, e^{-\frac{1}{2}(\varphi,-\Delta\varphi)}\, x_i^1 x_j^1 \mathcal{L} g(j, \frac{1}{2}|\varphi|^2) = \int d\varphi\, e^{-\frac{1}{2}(\varphi,-\Delta\varphi)}\, g(i, \frac{1}{2}|\varphi|^2) \tag{3.10}$$

Given the lemma, the theorem follows by choosing $g(i,\ell) = g_t(i,\ell) = \mathbb{E}_{i,\ell}(1_{X_t=j} g(L_t))$. Since

$$\frac{\partial}{\partial t} g_t = \mathcal{L} g_t \tag{3.11}$$

the left-hand side of the lemma can be written as

$$\frac{\partial}{\partial t}\left[-\sum_j \int d\varphi\, e^{-\frac{1}{2}(\varphi,-\Delta\varphi)}\, x_i^1 x_j^1 \mathbb{E}_j(1_{X_t=j} g(L_t + \frac{1}{2}|\varphi|^2))\right] \tag{3.12}$$

and the right-hand side gives

$$\int d\varphi\, e^{-\frac{1}{2}(\varphi,-\Delta\varphi)}\, \mathbb{E}_i(1_{X_t=j} g(L_t + \frac{1}{2}|\varphi|^2)). \tag{3.13}$$

The theorem now follows by integrating both sides over t using that the boundary term at $t \to \infty$ vanishes by assumption that g has rapid decay and that $\sum_j L_t^j = t \to \infty$.

Local Ward identities. The proof of the lemma relies on the general principle that symmetries of a measure imply constraints on expectations, known as Ward identities. Let T_j be the infinitesimal generator of translation in the 1-direction at vertex $j \in \Lambda$:

$$T_j = \frac{\partial}{\partial x_j^1}. \tag{3.14}$$

Since the Lebesgue measure is *translation invariant*, for any smooth f with sufficient decay,

$$\int (T_j f)(\varphi)\, d\varphi = 0. \tag{3.15}$$

In particular,

$$\int (T_j f)(\varphi) g(\varphi)\, e^{-H(\varphi)}\, d\varphi = \int f(\varphi)(-T_j g(\varphi) + (T_j H)(\varphi) g(\varphi))\, e^{-H(\varphi)}\, d\varphi \tag{3.16}$$

$$= \int f(\varphi)(T_j^* g)(\varphi)\, e^{-H(\varphi)}\, d\varphi \tag{3.17}$$

where T_j^* is the adjoint of T_j with respect to the measure $e^{-H}\, d\varphi$:

$$T_j^* g(\varphi) = -T_j g(\varphi) + (T_j H)(\varphi) g(\varphi). \tag{3.18}$$

Proof of Lemma 3.2. Let $H = \frac{1}{2}(\varphi, -\Delta_\beta \varphi)$. Then

$$-T_j H(\varphi) = (\Delta_\beta x^1)_j \tag{3.19}$$

and therefore

$$-T_j^* g(j, \frac{1}{2}|\varphi|^2) = (\Delta_\beta x^1)_j g(j, \frac{1}{2}|\varphi|^2) + x_j^1 \frac{\partial}{\partial \ell_j} g(j, \frac{1}{2}|\varphi|^2) \tag{3.20}$$

and by summation by parts

$$-\sum_j T_j^* g(j, \frac{1}{2}|\varphi|^2) = \sum_j x_j^1 \left[\Delta_\beta g(j, \frac{1}{2}|\varphi|^2) + \frac{\partial}{\partial \ell_j} g(j, \frac{1}{2}|\varphi|^2)\right] = \sum_j x_j^1 \mathcal{L} g(j, \frac{1}{2}|\varphi|^2). \tag{3.21}$$

The proof now follows from

$$\text{LHS} = \sum_j \int d\varphi\, e^{-H}\, x_j T_j^* g(j, \frac{1}{2}|\varphi|^2) = \sum_j \int d\varphi\, e^{-H}\, (T_j x_j) g(j, \frac{1}{2}|\varphi|^2) = \text{RHS} \tag{3.22}$$

using that $T_j x_i^1 = \delta_{ij}$.

□

3.3 Supersymmetric BFS–Dynkin isomorphism

The BFS–Dynkin isomorphism applies to fields taking values in the Euclidean space $\mathbb{R}^n$, where $n \geqslant 1$. It was observed by de Gennes and others that the limit '$n \to 0$' gives interesting results. It turns out that one way to make sense of '$n = 0$' is as $n = 2|2$. This means that we consider the case $n = 2$ and add two anticommuting fields, i.e., we consider the algebra $\Lambda^{2N}(\mathbb{R}^{2N})$. We write

$$\varphi = (x, y, \xi, \eta) \tag{3.23}$$

with complex coordinates $(z, \bar{z}, \zeta, \bar{\zeta})$. Thus x, y and ξ, η are canonical even and odd generators of $\Lambda^{2N}(\mathbb{R}^{2N})$. We set

$$\varphi_i \cdot \varphi_j = x_i x_j + y_i y_j - \xi_i \eta_j + \eta_i \xi_j = \frac{1}{2}\Big(z_i \bar{z}_j + z_j \bar{z}_i + \zeta_i \bar{\zeta}_j + \zeta_j \bar{\zeta}_i\Big). \tag{3.24}$$

The supersymmetric Gaussian free field, or free field with target space $\mathbb{R}^{2|2}$, is the form

$$e^{-\frac{1}{2}(\varphi, -\Delta_\beta \varphi) - \frac{1}{2}m^2(\varphi,\varphi)} \in \Lambda^{2N}(\mathbb{R}^{2N}). \tag{3.25}$$

Theorem 3.3. *For $g : \mathbb{R}^N \to \mathbb{R}$ smooth with rapid decay,*

$$\int e^{-\frac{1}{2}(\varphi, -\Delta_\beta \varphi)}\, g(\frac{1}{2}\varphi \cdot \varphi)\, x_i x_j = \int e^{-\frac{1}{2}(\varphi, -\Delta_\beta \varphi)} \int_0^\infty \mathbb{E}_i(1_{X_t = j}\, g(\frac{1}{2}\varphi \cdot \varphi + L_t))\, dt. \tag{3.26}$$

Proof. The proof is identical to the non-supersymmetric one for $\mathbb{R}^2$, which only relies on translation invariance in the x-direction. □

Now note that the right-hand side only depends on $\varphi_i \cdot \varphi_j$ and that $\varphi_i \cdot \varphi_j$ is supersymmetric and localizes at 0, i.e.,

$$Q(\varphi_i \cdot \varphi_j) = 0, \quad (\varphi_i \cdot \varphi_j)|_0(0) = 0. \tag{3.27}$$

Thus we can apply the Localisation Theorem and the right-hand side simplifies to

$$\int_0^\infty \mathbb{E}_i(1_{X_t = j}\, g(L_t))\, dt. \tag{3.28}$$

Thus both sides are completely decoupled!

3.4 Applications

Random walk representations are known to be an important tool in the study of spin systems and quantum field theory. Conversely field theory, in particular with supersymmetry, can be used to understand random walks.

As an example, take

$$g(t) = e^{-\sum_i (\lambda t_i^2 - \nu t_i)} \quad (\lambda > 0, \nu \in \mathbb{R}). \tag{3.29}$$

With this choice of g, the BFS–Dynkin isomorphism provides a representation for the two-point function of the n-component $|\varphi|^4$-model, which has density proportional to

$$\exp\Big(-\frac{1}{2}(\varphi, -\Delta\varphi) - \sum_i \Big(\frac{1}{4}\lambda|\varphi_i|^4 + \frac{1}{2}\nu|\varphi_i|^2\Big)\Big). \tag{3.30}$$

In the supersymmetric case, it gives the *weakly self-avoiding walk* or *Edwards model* which has density with respect to the measure induced by simple random walk on paths of length t proportional to

$$\exp\Big(-\sum_i \Big(\lambda (L_t^i)^2 + \nu L_t^i\Big)\Big) = \exp\Big(-\lambda \int_0^t \int_0^t ds_1\, ds_2\, 1_{X_{s_1} = X_{s_2}} - \nu t\Big). \tag{3.31}$$

The integral in the exponent on the right-hand side is known as self-intersection local time.

As simple illustration how random walks can be used to study field is the following exercise.

Exercise 3.4. *Use the BFS–Dynkin isomorphism to show that the two-point function of the $|\varphi|^4$ model on $\mathbb{Z}^d$ decays exponentially whenever $\nu > 0$, i.e.,*

$$\langle \varphi_i^1 \varphi_j^1 \rangle_{g,\nu} \leqslant C e^{-c|i-j|}. \tag{3.32}$$

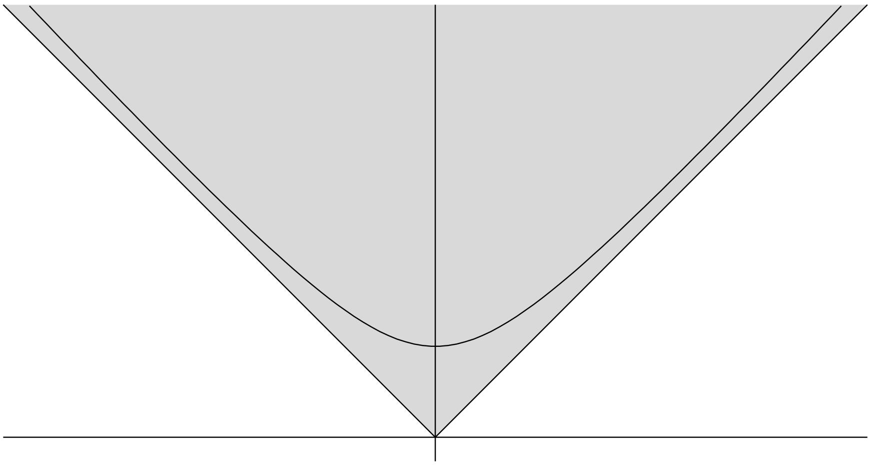

FIGURE 4.1. Minkowski space $\mathbb{R}^{n+1}$. The shaded area is the causal future and the hyperboloid is $\mathbb{H}^n$.

4 Hyperbolic symmetry

4.1 Hyperbolic space

The Gaussian free field is the sigma model associated to the Euclidean space $\mathbb{R}^n$. We now consider an analogue for hyperbolic space $\mathbb{H}^n$. Minkowski space is $\mathbb{R}^{n+1}$ with the indefinite inner product

$$u_1 \cdot u_2 = x_1 \cdot x_2 - z_1 z_2 = \sum_{\alpha=1}^{n} x_1^\alpha x_2^\alpha - z_1 z_2 \tag{4.1}$$

where

$$u = (x^1, \ldots, x^n, z). \tag{4.2}$$

The points $u \in \mathbb{R}^{n+1}$ with $u \cdot u < 0$ are *time-like*. Hyperbolic space $\mathbb{H}^n$ can be defined as

$$\mathbb{H}^n = \{u = (x, z) = (x^1, \ldots, x^n, z) \in \mathbb{R}^{n+1} : u \cdot u = -1, z > 0\}. \tag{4.3}$$

Thus $\mathbb{H}^n$ is parameterized by $x \in \mathbb{R}^n$ with $z = \sqrt{1 + |x|^2}$.

The group preserving the Minkowski inner product is the Lorentz group $O(n, 1)$. The restricted Lorentz group $SO^+(n, 1)$ is the subgroup of $T \in O(n, 1)$ with $\det T = 1$ and $T_{n+1,n+1} > 0$. It acts on the causal future, i.e., the interior of the forward light cone (see Figure 4.1). The elements of $SO^+(n, 1)$ are compositions of rotations and boosts. For example, for $n = 2$,

$$R_\theta = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad T_\alpha = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh\alpha & \sinh\alpha \\ 0 & \sinh\alpha & \cosh\alpha \end{pmatrix} \tag{4.4}$$

are a rotations in the x^1x^2-plane and a boost in the x^2z-plane. The infinitesimal versions of this rotation and boost are

$$R = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \tag{4.5}$$

Hyperbolic space $\mathbb{H}^n$ is the orbit under $SO^+(n, 1)$ of the point $e = (0, 1)$. The $SO^+(n, 1)$-invariant measure on $\mathbb{H}^n$ can be written as

$$du = \frac{dx^1 \ldots dx^n}{z(x^1, \ldots x^n)}, \quad z(x^1, \ldots, x^n) = \sqrt{1 + (x^1)^2 + \cdots + (x^n)^2}. \tag{4.6}$$

Exercise 4.1. *Let $F = \{u = (x,z) \in \mathbb{R}^{n+1} : u \cdot u < 0 \text{ and } z > 0\}$ be the causal future. Let $SO^+(n,1)$ be the group of matrices T with $\det T = 1$ such that $Tu \cdot Tu = u \cdot u$ for all u and $TF \subset F$. Show that for all $g : \mathbb{R}^{n+1} \to \mathbb{R}$ with compact support in F,*

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}} g(u) f(u \cdot u)\, dz\, dx = \int_{\mathbb{R}^n} \int_{\mathbb{R}} g(Tu) f(u \cdot u)\, dz\, dx. \tag{4.7}$$

Show further that

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}} g(x,z) \delta_\varepsilon(|x|^2 - z^2 + t)\, dz\, dx \to \int_{\mathbb{R}^n} g(x,z) \frac{dx}{2z}, \quad z = \sqrt{t + |x|^2}. \tag{4.8}$$

Indeed, consider only the z integral and abbreviate $a = |x|^2 + t$. We can change variable from z to $w = z^2 - a^2$. Then $dw = 2zdz = 2\sqrt{w + a^2} dz$ so $dz = dw/(2\sqrt{w + a^2})$. Then taking $\varepsilon \to 0$ the w integral gets replaced by $w = 0$.

4.2 Hyperbolic sigma model and vertex-reinforced jump process

For $u \in (\mathbb{H}^n)^\Lambda$, we write

$$\frac{1}{2}(u, -\Delta_\beta u) = \frac{1}{2} \sum_{i,j} (-\Delta_\beta)_{ij} (u_i \cdot u_j) = \frac{1}{2} \sum_{i,j} \beta_{ij} (-1 - u_i \cdot u_j), \tag{4.9}$$

similarly as before except that now the inner product is the Minkowski inner product. We fix any $e \in \mathbb{H}^n$. The hyperbolic sigma model with target space $\mathbb{H}^n$ is the probability measure

$$e^{-\frac{1}{4}(u, -\Delta_\beta u) - h(u,e)}\, du. \tag{4.10}$$

By symmetry, we may assume that $e = (0,0,1)$ so that $(u,e) = z$.

In the way that the free field is related to simple random walks, the hyperbolic sigma model is intimately related to a class of linearly *reinforced walks*. These are history dependent processes defined as follows. The vertex reinforced jump process (VJRP) has transition probabilities

$$\mathbb{P}(X_{t+\delta t} = j | X_t = i, L_t^j = \ell_j) = \beta_{ij}(1 + \ell_j)\delta t + o(\delta t). \tag{4.11}$$

The process (X_t) is not a Markov process, but the joint process (X_t, L_t) is. Its generator acts on functions $g : \Lambda \times \mathbb{R}^\Lambda \to \mathbb{R}$ by

$$\mathcal{L} g(i, \ell) = \sum_j \beta_{ij} (1 + \ell_j)(g(j,\ell) - g(i,\ell)) + \frac{\partial}{\partial \ell_i} g(i,\ell). \tag{4.12}$$

We write $\mathbb{E}_{i,\ell}$ to denote the expectation of this process with initial condition $(X_0, L_0) = (i, \ell)$.

Theorem 4.2 (BHS [2])**.** *For every smooth $g : \mathbb{R}^\Lambda \to \mathbb{R}$ with sufficient decay,*

$$\int_{\mathbb{H}^n} du\, e^{-\frac{1}{2}(u,-\Delta u)} x_i^1 x_j^1 g(z-1) = \int_{\mathbb{H}^n} du\, e^{-\frac{1}{2}(u,-\Delta u)} z_i \int_0^\infty dt\, \mathbb{E}_{i,z-1}(1_{X_t = j} g(L_t)). \tag{4.13}$$

The proof again follows from an equivalent statement for the generator for the walk, given in the next lemma.

Lemma 4.3. *For all smooth $g : \Lambda \times \mathbb{R}^\Lambda \to \mathbb{R}$ with sufficient decay,*

$$-\sum_j \int_{\mathbb{H}^n} du\, e^{-\frac{1}{2}(u,-\Delta u)} x_i^1 x_j^1 \mathcal{L} g(j, z-1) = \int_{\mathbb{H}^n} du\, e^{-\frac{1}{2}(u,-\Delta u)} z_i g(i, z-1) \tag{4.14}$$

To prove the lemma, we proceed as for the simple random walk, except that now T_j is the infinitesimal generator of the *Lorentz boost* in the x^1z-plane at vertex $j \in \Lambda$. In the parametrization of $\mathbb{H}^n$ by $x \in \mathbb{R}^n$, it is given by

$$T_j = z_j \frac{\partial}{\partial x_j^1}. \tag{4.15}$$

Indeed, $T_j x^i = 0$ for $i \geqslant 2$ and

$$T_j x_j^1 = z_j, \quad T_j z_j = x_j^1. \tag{4.16}$$

By construction, the measure du on $\mathbb{H}^n$ is Lorentz invariant so that we obtain a local Ward identity exactly as in the flat case. Next we again compute the adjoint of T_j with respect to the measure of the hyperbolic sigma model. Let

$$H = \frac{1}{2}(u, -\Delta_\beta u) = \frac{1}{2}\sum_{j,k} \beta_{jk}(-u_j \cdot u_k - 1) = \frac{1}{2}\sum_{j,k} \beta_{jk}\Big(z_j z_k - \sum_{\alpha=1}^{n} x_j^\alpha x_k^\alpha - 1\Big). \tag{4.17}$$

Then

$$T_j H = \sum_k \beta_{jk}(x_j^1 z_k - z_j x_k^1). \tag{4.18}$$

Therefore

$$\begin{aligned} -T_j^* g(j, z-1) &= (-(T_j H) + T_j) g(j, z-g)) \\ &= \sum_k \beta_{jk}(x_k^1 z_j - x_j^1 z_k) g(j, z-1) + x_j^1 \frac{\partial}{\partial \ell_j} g(j, z-1) \end{aligned} \tag{4.19}$$

$$\begin{aligned} -\sum_j T_j^* g(j, z-1) &= \sum_j x_j^1 \left[\beta_{jk} z_k (g(k, z-1) - g(j, z-1)) + \frac{\partial}{\partial \ell_j} g(j, z-1)\right] \\ &= \sum_j x_j^1 \mathcal{L} g(j, z-1). \end{aligned} \tag{4.20}$$

The proof now again follows from

$$\text{LHS} = \sum_j \int du\, e^{-H}\, x_j T_j^* g(j, \frac{1}{2}|\varphi|^2) = \sum_j \int du\, e^{-H}\, (T_j x_j) g(j, \frac{1}{2}|\varphi|^2) = \text{RHS} \tag{4.21}$$

using that $T_j x_i^1 = z_i \delta_{ij}$.

4.3 Hyperbolic superplane

The algebra of the hyperbolic superplane $\Lambda^2(\mathbb{H}^2)$ is defined as follows. Let ξ, η be generators of the Grassmann algebra Λ^2. We write u^0 for point in $\mathbb{H}^2$ and denote its components by $u^0 = (x, y, z^0) \in \mathbb{H}^2$. The superscript 0 distinguishes it from the supersymmetric version that we introduce next. Set

$$z = \sqrt{1 + x^2 + y^2 - 2\xi\eta}. \tag{4.22}$$

Thus z is an even element of $\Lambda^2(\mathbb{H}^2)$ with 0-degree part z^0 and $u = (x, y, z, \xi, \eta)$ obeys the constraint $u \cdot u = -1$ where we define now

$$u \cdot u = x^2 + y^2 - z^2 - 2\xi\eta. \tag{4.23}$$

The integral of a form $F \in \Lambda^2(\mathbb{H}^2)$ is now defined by

$$\int_{\mathbb{H}^{2|2}} F(u) = \frac{1}{2\pi} \int_{\mathbb{R}^2} dx\, dy\, \partial_\eta\, \partial_\xi\, \frac{1}{z} F(x, y, z, \eta, \xi). \tag{4.24}$$

Exercise 4.4. *Let x, y, z and ξ, η be even and odd generators for $\Lambda^2(\mathbb{R}^3)$. Show that*

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}} dx\, dy\, dz\, \partial_\eta\, \partial_\xi\, g(x,y,\xi,\eta,z)\delta_\varepsilon(|x|^2+2\xi\eta-z^2+t)\, dz\, dx \to \int_{\mathbb{R}^n} dx\, dy\, \partial_\eta\, \partial_\xi \frac{1}{2z} g(x,y,\xi,\eta,z), \tag{4.25}$$

where on the right-hand side $z = \sqrt{t+x^2+y^2-2\xi\eta}$.

The infinitesimal generator of the Lorentz boost is now defined to be $T = z\frac{\partial}{\partial x}$ acting on $\Lambda^2(\mathbb{R}^2)$. It is again elementary to verify that

$$Tx = z, \quad Tz = x, \quad Ty = 0, \quad T\xi = 0, \quad T\eta = 0, \tag{4.26}$$

and that the integral defined above is invariant under T, i.e., for smooth $F \in \Lambda^2(\mathbb{R}^2)$ with sufficient decay,

$$\int_{\mathbb{H}^{2|2}} TF = 0. \tag{4.27}$$

4.4 Supersymmetric hyperbolic sigma model

The supersymmetric hyperbolic sigma model is defined by the form

$$e^{-(u,-\Delta_\beta u)-h(u,e)} \tag{4.28}$$

integrated with respect to the integral of the hyperbolic superplane.

Theorem 4.5. *For every smooth $g : \Lambda \times \mathbb{R}^\Lambda \to \mathbb{R}$ with sufficient decay,*

$$\int_{(\mathbb{H}^{2|2})^\Lambda} e^{-\frac{1}{2}(u,-\Delta u)} x_i^1 x_j^1 g(z-1) = \int_0^\infty \mathbb{E}_{i,0}(1_{X_t=j}\, g(L_t)). \tag{4.29}$$

In particular,

$$\langle x_i x_j\rangle_{\beta,h} = \int_0^\infty \mathbb{E}_{i,0}(1_{X_t=j})e^{-ht}. \tag{4.30}$$

Remark 4.6. *The vertex-reinforced jump process was introduced from probabilistic motivation. Results for $\mathbb{Z}^d$ include (all proved using supersymmetry):*

- *For $d \geqslant 3$ and $\beta \gg 1$: $\langle x_i x_j\rangle_{\beta,h}$ is bounded and quasi-diffusive [6] and, as a consequence of this, the VRJP is quasi-diffusive [2, 9, 10].*
- *For $d \geqslant 1$ and $\beta \ll 1$: $\langle x_i x_j\rangle_{\beta,h} \leqslant Ch^{-1}e^{-c|i-j|}$ which implies that $\langle x_i^2\rangle \geqslant ch^{-1}$ is unbounded [5], and that the VRJP is localized [2, 9].*
- *For $d = 2$ and any $\beta > 0$, $\langle x_0^2\rangle_{\beta,h}$ is unbounded as $h \downarrow 0$ and hence the VRJP is recurrent [2].*

This behaviour is consistent with that expected picture for the Anderson transition of random Schrödinger operators and random band matrices. In $d = 2$, it is further expected that $\langle x_i x_j\rangle_{\beta,h}$ decays exponentially uniformly in $h > 0$ for all $\beta > 0$.

4.5 Horospherical coordinates

Non-supersymmetric case. Aside from the coordinates $x \in \mathbb{R}^n$ for $\mathbb{H}^n$ that we have used so far, there are other very useful coordinates for $\mathbb{H}^n$. For notational convenience we restrict to $\mathbb{H}^2$, though nothing is very different for $\mathbb{H}^n$. Horospherical coordinates for $\mathbb{H}^2$ are given by $(s,t) \mapsto (x,y)$ where

$$x = \sinh t - \frac{1}{2}s^2 e^t, \quad y = e^t s. \tag{4.31}$$

This implies

$$z = \sqrt{1+x^2+y^2} = \cosh t + \frac{1}{2}s^2 e^t \tag{4.32}$$

and the Jacobian of this change of variable is

$$J(s,t) = \det \begin{pmatrix} \cosh t - \frac{1}{2}s^2 e^t & -se^t \\ se^t & e^t \end{pmatrix} = e^t z. \tag{4.33}$$

Therefore

$$\int_{\mathbb{H}^2} F(x,y,z) = \int dx\, dy\, \frac{1}{z(x,y)}\, F(x,y,z(x,y)) = \int dt\, ds\, e^t\, F(x(s,t),y(s,t),z(s,t)). \tag{4.34}$$

Supersymmetric case. Horospherical coordinates for $\mathbb{H}^{2|2}$ are defined as follows. Let t, s and $\psi, \bar{\psi}$ be the standard even and odd generators of $\Lambda^2(\mathbb{R}^2)$. Set

$$x = \sinh t - (\frac{1}{2}s^2 + \psi\bar{\psi})e^t \tag{4.35}$$

$$z = \cosh t + (\frac{1}{2}s^2 + \psi\bar{\psi})e^t \tag{4.36}$$

$$y = se^t \tag{4.37}$$

$$\xi = \psi e^t \tag{4.38}$$

$$\eta = \bar{\psi} e^t \tag{4.39}$$

Lemma 4.7. *The x, y, ξ, η are again generators for $\Lambda^2(\mathbb{R}^2)$ and $u\cdot u = -1$ with notation used earlier. The following change of variable formula holds:*

$$\int \frac{dx\, dy\, \partial_\eta \partial_\xi}{2\pi} \frac{1}{z(x,y,\xi,\eta)} F(x,y,\xi,\eta) = \int \frac{dt\, ds\, \partial_\psi \partial_{\bar{\psi}}}{2\pi} e^{-t}\, F(x(t,s,\psi,\bar{\psi}), e^t s, e^t\psi, e^t\bar{\psi}), \tag{4.40}$$

where

$$z(x,y,\xi,\eta) = \sqrt{1+x^2+y^2-2\xi\eta}. \tag{4.41}$$

The lemma can be derived from a general change of variable formula for superintegrals that can be found in [3], but we here verify it by hand. This an example of how such change of variables can mix components of different degrees.

Proof. We suppress the argument y in the following. By Example 2.14 with F replaced by F/z,

$$\int dx_0\, \partial_\eta\, \partial_\xi\, F(x_0,\xi,\eta) = \int dx_0\, \partial_\eta\, \partial_\xi\, F(x_0 + g(x_0)\xi\eta, \xi, \eta) \frac{1}{z(x_0 + g(x_0)\xi\eta, \xi, \eta)} (1 + g'(x_0)\xi\eta). \tag{4.42}$$

Write $z_0 = z(x_0, y_0, 0, 0)$ and take $g(x_0) = 1/(x_0 + z_0) = 1/(x + z)$. We will verify that

$$\frac{1}{z(x_0 + g(x_0)\xi\eta, y, \xi, \eta)} = \frac{1}{z_0}\left(1 + \xi\eta \frac{1}{z_0(x_0+z_0)}\right), \qquad 1 + g'(x_0)\xi\eta = 1 - \xi\eta\frac{1}{z_0(x_0+z_0)}. \tag{4.43}$$

and hence that the product of these two expressions equals $1/z_0$. This gives

$$\int_{\mathbb{R}^2} dx_0\, dy_0\, \partial_\eta\, \partial_\xi\, F(x_0, y_0, \xi, \eta) \frac{1}{z(x_0, y_0, \xi, \eta)} = \int_{\mathbb{R}^2} dx_0\, dy_0\, \partial_\eta\, \partial_\xi\, \frac{1}{z_0}\, F(x_0 + \frac{\xi\eta}{x_0+z_0}, y_0, \xi, \eta) \tag{4.44}$$

Next we use horospherical coordinates for $\mathbb{H}^2$ and that $x_0 + z_0 = e^t$ to write the right-hand side as

$$\int dt\, ds\, \partial_\eta\, \partial_\xi\, e^t\, F(\sinh t - (\frac{1}{2}s^2 - e^{-2t}\xi\eta)e^t, se^t, \xi, \eta)$$
$$= \int dt\, ds\, \partial_\xi\, \partial_\eta\, e^t\, F(\sinh t - (\frac{1}{2}s^2 + e^{-2t}\xi\eta)e^t, se^t, \xi, \eta). \tag{4.45}$$

Then with $\psi = e^{-t}\xi$ and $\bar\psi = e^{-t}\eta$, this is the claim

$$\int dt\, ds\, \partial_\psi\, \partial_{\bar\psi}\, e^{-t}\, F(\sinh t - (\frac{1}{2}s^2 + \psi\bar\psi)e^t, se^t, e^t\psi, e^t\bar\psi). \tag{4.46}$$

Finally, it remains to verify (4.43):

$$\begin{aligned}
\frac{1}{z(x+\frac{\xi\eta}{x+z}, y, \xi, \eta)} &= \left((x+\frac{\xi\eta}{x+z})^2 + y^2 - 2\xi\eta - 1\right)^{-1/2} \\
&= \left(x^2 + y^2 - 2\xi\eta(1-\frac{x}{x+z}) - 1\right)^{-1/2} \\
&= \left(x^2 + y^2 - 2\xi\eta\frac{z}{x+z} - 1\right)^{-1/2} \\
&= \frac{1}{z(x,y,0,0)}\left(1 + \frac{1}{z(x,y,0,0)^2}\xi\eta\frac{z}{x+z}\right) \\
&= \frac{1}{z_0}\left(1 + \xi\eta\frac{1}{z_0(x+z_0)}\right)
\end{aligned} \tag{4.47}$$

whereas the factor $1 + g'(x)\xi\eta$ gives

$$1 + \xi\eta\partial_x\frac{1}{x+z} = 1 - \xi\eta\frac{1+\frac{x}{z}}{(x+z)^2} = 1 - \xi\eta\frac{1}{z(x+z)} = 1 - \xi\eta\frac{1}{z_0(x_0+z_0)} \tag{4.48}$$

as needed. □

4.6 Hyperbolic sigma model in horospherical coordinates

In horospherical coordinates,

$$\begin{aligned}
(u, -\Delta_\beta u) &= \sum_{i,j}\beta_{ij}(-u_i\cdot u_j - 1) && (4.49)\\
&= \sum_{i,j}\beta_{ij}\left(\cosh(t_i - t_j) - 1 + (\frac{1}{2}(s_i - s_j)^2 + (\psi_i - \psi_j)(\bar\psi_i - \bar\psi_j))e^{t_i+t_j}\right) && (4.50)
\end{aligned}$$

In particular, the right-hand side is quadratic in s and the odd variables. In particular, for observables that only depend on s and t, the ψ and $\bar\psi$ can be integrated out.

Exercise 4.8. *Let*

$$\beta_{ij}(t) = \beta_{ij}e^{t_i+t_j}, \quad h_i(t) = he^{t_i}. \tag{4.51}$$

Show that then

$$\partial_\psi\partial_{\bar\psi}\exp\left(-\frac{1}{2}\sum_{i,j}(\psi_i - \psi_j)(\bar\psi_i - \bar\psi_j))e^{t_i+t_j} + \sum_j h\bar\psi_i\psi_i e^{t_i}\right) = \det(-\Delta_{\beta(t)} + h(t)). \tag{4.52}$$

Therefore

$$\int e^{-(u,-\Delta_\beta u) - \sum_i hz_i} f(x+z, y) = \int dt\, ds\, e^{-t}e^{-\tilde H(t,s)}f(e^t, se^t) \tag{4.53}$$

with

$$\tilde H(t,s) = ... - \log\det(-\Delta_{\beta(t)} + h(t)) \tag{4.54}$$

Thus the supersymmetric expectation of any function of $e^t = x+z$ and $y = se^t$ can be computed in terms of a *probability measure*. Clearly, $T_j = \frac{\partial}{\partial s_j}$ is a symmetry of the measures on $\mathbb{H}^n$ and $\mathbb{H}^{2|2}$. In fact, it is a combination of a Lorentz boost and a rotation. Moreover,

$$\frac{\partial^2}{\partial s_j^2} z_j = e^{t_j} = x_j + z_j, \qquad \frac{\partial^2}{\partial s_i \partial s_l}(-1 - u_j \cdot u_l) = \begin{cases} -e^{t_j+t_l} = -(x_j+z_j)(x_l+z_l), & i=j, \\ +e^{t_j+t_l} = +(x_j+z_j)(x_l+z_l), & i=l, \\ 0, & \text{else.} \end{cases} \tag{4.55}$$

4.7 Application: Mermin–Wagner theorem

In this section we assume that $\Lambda = \Lambda_L$ is the discrete d-dimensional torus $\mathbb{Z}^d/(L\mathbb{Z})^d$ of side length $L \in \mathbb{N}$, and that β is translation invariant and finite range. For simplicity, take $\beta_{ij} = \beta 1_{i\sim j}$. Denote

$$\lambda(p) := \sum_{j\in\Lambda} \beta_{0j}(1 - \cos(p \cdot j)), \qquad p \in \Lambda^\star, \tag{4.56}$$

where here $\cdot$ is the Euclidean inner product on $\mathbb{R}^d$ and $\Lambda^\star$ is the Fourier dual of the discrete torus Λ. Denote the two-point function and its Fourier transform by

$$G_{\beta,h}(j) = G^L_{\beta,h}(j) := \langle y_0 y_j \rangle_{\beta,h}, \qquad \hat{G}_{\beta,h}(p) = \hat{G}^L_{\beta,h}(p) = \sum_{j\in\Lambda} G_{\beta,h}(j) e^{i(p\cdot j)}. \tag{4.57}$$

The following theorem is an analogue of the Mermin–Wagner Theorem.

Theorem 4.9. *Let $\Lambda = \mathbb{Z}^d/(L\mathbb{Z})^d$, $L \in \mathbb{N}$. For the $\mathbb{H}^n$ model, $n \geqslant 2$, with magnetic field $h > 0$,*

$$\hat{G}_{\beta,h}(p) \geqslant \frac{1}{(1 + (n+1)G_{\beta,h}(0))\lambda(p) + h}. \tag{4.58}$$

Similarly, for the $\mathbb{H}^{2|2}$ model with $h > 0$,

$$\hat{G}_{\beta,h}(p) \geqslant \frac{1}{(1 + G_{\beta,h}(0))\lambda(p) + h}. \tag{4.59}$$

In $d \leqslant 2$, these inequalities imply that $G(0)$ diverges as $L \to \infty$ and then $h \downarrow 0$. Indeed, since $(2\pi L)^{-d} \sum_{p\in\Lambda^*} e^{i(p\cdot j)} = 1_{j=0}$, summing the bounds (4.58) and (4.59) over $p \in \Lambda^\star$ and interchanging sums implies (with $n = 0$ for $\mathbb{H}^{2|2}$)

$$G_{\beta,h}(0) \geqslant \frac{1}{(2\pi L)^d} \sum_{p\in\Lambda^\star} \frac{1}{(1 + (n+1)G_{\beta,h}(0))\lambda(p) + h}. \tag{4.60}$$

The assumption of β being finite range and non-negative implies $\lambda(p) \leqslant C(\beta)|p|^2$. If $d \leqslant 2$ it follows that

$$\lim_{L\to\infty} \frac{1}{(2\pi L)^d} \sum_{p\in\Lambda^\star} \frac{1}{\lambda(p) + h} \uparrow \infty \quad \text{as } h \downarrow 0. \tag{4.61}$$

In particular, the VRJP spends an infinite amount of time at the origin.

Proof of (4.59). We use that the expectation of a function $F(y)$ can be written using horospherical coordinates in terms of the *probability measure* with density $e^{-\tilde{H}}\, ds\, dt$. Throughout this proof, we denote the expectation with respect to this probability measure by $\langle\cdot\rangle$. Let

$$S(p) = \frac{1}{\sqrt{|\Lambda|}} \sum_j e^{ip\cdot j} y_j, \qquad T(p) = \frac{1}{\sqrt{|\Lambda|}} \sum_j e^{-ip\cdot j} \frac{\partial}{\partial s_j}, \tag{4.62}$$

By the Cauchy–Schwarz inequality, and since $S(p)$ is a function of the y,

$$\langle |S(p)|^2 \rangle = \langle |S(p)|^2 \rangle \geqslant \frac{|\langle S(p)T(p)\tilde{H}\rangle|^2}{\langle |T(p)\tilde{H}|^2 \rangle} = \frac{|\langle S(p)T(p)\tilde{H}\rangle|^2}{\langle T(p)\overline{T(p)}\tilde{H}\rangle}. \tag{4.63}$$

where the last equality follows from the fact that the probability measure $\langle \cdot \rangle$ obeys the integration by parts $\langle FT(p)\tilde{H}\rangle = \langle T(p)F\rangle$ identity.

Therefore by translation invariance we find that, as in the case of $\mathbb{H}^n$,

$$\langle |S(p)|^2 \rangle = \frac{1}{|\Lambda|}\sum_{j,l} e^{ip\cdot(j-l)}\langle y_j y_l\rangle = \frac{1}{|\Lambda|}\sum_{j,l} e^{ip\cdot(j-l)}\langle y_0 y_{j-l}\rangle = \sum_j e^{i(p\cdot j)}\langle y_0 y_j\rangle, \tag{4.64}$$

$$\langle S(p)T(p)\tilde{H}\rangle = \langle T(p)S(p)\rangle = \frac{1}{|\Lambda|}\sum_{j,l} e^{ip\cdot(j-l)}\langle \frac{\partial y_j}{\partial s_l}\rangle = \frac{1}{|\Lambda|}\sum_j \langle e^{t_j}\rangle = \frac{1}{|\Lambda|}\sum_j \langle x_j + z_j\rangle = 1. \tag{4.65}$$

Since $e^t = x + z$, by localization, Cauchy–Schwarz, and translation invariance we have

$$\langle e^{t_j + t_l}\rangle = 1 + \langle y_j y_l\rangle \leqslant 1 + \langle y_0^2\rangle. \tag{4.66}$$

Using (4.66) and the integration by parts identity it follows that

$$\begin{aligned}\langle |T(p)\tilde{H}|^2\rangle = \langle T(p)\overline{T(p)}\tilde{H}\rangle &= \frac{1}{|\Lambda|}\sum_{j,l}\beta_{jl}\langle e^{t_j+t_l}\rangle(1-\cos(p\cdot(j-l))) + \frac{h}{|\Lambda|}\sum_j \langle e^{t_j}\rangle \\ &\leqslant \frac{1}{|\Lambda|}\sum_{j,l}\beta_{jl}(1+\langle y_0^2\rangle)(1-\cos(p\cdot(j-l))) + h \\ &= (1+\langle y_0^2\rangle)\lambda(p) + h. \end{aligned} \tag{4.67}$$

In summary, we have proved

$$\sum_j e^{i(p\cdot j)}\langle y_0 y_j\rangle = \langle |S(p)|^2\rangle \geqslant \frac{|\langle S(p)T(p)\tilde{H}\rangle|^2}{\langle |T(p)\tilde{H}|^2\rangle} \geqslant \frac{1}{(1+\langle y_0^2\rangle)\lambda(p) + h} \tag{4.68}$$

as claimed. □

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