Probability Theory II (Spring 2026)

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Primary reference: Varadhan, Stochastic Processes
Prerequisites: Varadhan, Probability Theory
Other references:
Norris, Advanced Probability
Stroock, Probability Theory, An Analytic View
Le Gall, Brownian Motion, Martingales, and Stochstic Calculus

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1. Continuous-time processes & martingales

From now on, $(\Omega, \mathcal{F}, P)$ denotes a probability space on which all random variables are defined.

1.1. Continuous-time processes

Defn. Given an index set $\mathbb{T}$, a stochastic process $X = \{X_t : t \in \mathbb{T}\}$ is a collection of random variables $X_t = X(t, \omega)$, $t \in \mathbb{T}$, $\omega \in \Omega$, taking values in some measurable space.

- If $\mathbb{T}$ is a countable set, X is a discrete-time stochastic process.
- If $\mathbb{T} \subseteq \mathbb{R}$ is a finite or infinite interval, X is a continuous-time stochastic process.

Careful: For $\mathbb{T}$ uncountable, events like

$$\{\omega : X_t(\omega) \in [a,b] \ \forall t \in \mathbb{T}\}$$

or

$$\{\omega : t \mapsto X_t(\omega) \text{ is continuous}\}$$

are not measurable in general.

Soln: Build regularity into Ω.

Defn. $C[a,b]$ is the Banach space of continuous functions $f:[a,b]\to\mathbb{R}$ with norm

$$\|f\| = \sup_{t\in[a,b]} |f(t)|.$$

Defn. $D[a,b]$ is the Polish space (complete sep metric) of left-continuous functions with right limits (càdlàg functions) $f:[a,b]\to\mathbb{R}$ with metric

$$d(f,g) = \inf\Big\{ r>0 : \exists\, \lambda:[a,b]\to[a,b] \text{ increasing, invertible, and continuous s.t. } \sup_t |\lambda(t)-t| \le r,\ \sup_t |f(t)-g(\lambda(t))| \le r \Big\}.$$

Prop. $D[a,b]$ is indeed a Polish space.
[Billingsley, Convergence of Prob. Measures, Chapter 3].

Prop. The σ-fields of Borel sets of $C[a,b]$ or $D[a,b]$ are generated by the cylinder sets, i.e., by

$$\{x\in X : \pi_t x = x(t)\in I\},\quad I\in\mathcal{B}(\mathbb{R}),\ t\in[a,b]$$

Sketch. Let $\mathcal{F}$ denote the cylinder σ-field. Since $\{x\in X : x(t)\in I\}$ is a Borel set, $\mathcal{F}\subset\mathcal{B}$. Conversely,

$$\{x\in C : \|x\|\le r\} = \bigcap_{t\in[a,b]\cap\mathbb{Q}} \pi_t^{-1}([-r,+r]) \in \mathcal{F}.$$

Exercise: $X = D$

We will always use the cylinder σ-field (= Borel σ-field) on $C[a,b]$ or $D[a,b]$. The space $C[0,\infty)$ is defined by $x \in C[0,\infty)$ if $x|_{[0,T]} \in C[0,T]$ and likewise for $D[0,\infty)$ with unif. on compacts metric.

<u>Defn</u>. Let $(\Omega, \mathcal{F})$ be a measurable space. Then a family $\{\mathcal{F}_t : t \geq 0\}$ of sub-σ-fields $\mathcal{F}_t \subset \mathcal{F}$ is

- a <u>filtration</u> if $\mathcal{F}_s \subseteq \mathcal{F}_t$ if $s \leq t$.
- a <u>right-cont</u>. filtration if $\mathcal{F}_t \supset \mathcal{F}_{t+} = \bigcap_{s>t} \mathcal{F}_s$

<u>Defn</u>. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t))$ be a filtered measurable space. Then a stochastic process (X_t) is

- <u>adapted</u> if $\forall t : X_t \in \mathcal{F}_t$, i.e., X_t is $\mathcal{F}_t$-meas.
- <u>integrable</u> if $\forall t : X_t \in L^1$, i.e., $E|X_t| < \infty$.

<u>Example</u>. If Ω is a space of functions on $\mathbb{T} \subset [0,\infty)$,

$$\mathcal{F}_t = \sigma(\omega(s) : s \leq t)$$

is the <u>canonical filtration</u>. The <u>canonical process</u> $X_t(\omega) = \omega(t)$, $\omega \in \Omega$ is $(\mathcal{F}_t)$-adapted.

If Ω is a space of right-cont. functions, it is also adapted to $(\mathcal{F}_{t+})$.

Prop. Let (X_t) be a $D[0,\infty)$-valued adapted process. Then (X_t) is <u>progressively measurable</u>: $\forall T>0$

$$(\omega,t) \mapsto X_t(\omega)$$

on $\Omega\times[0,T]$ is measurable w.r.t. $\mathcal{F}_T \otimes \mathcal{B}([0,T])$.

<u>Sketch</u>. By right-cont. can approximate by

$$X_t^n(\omega) = X_{2^{-n}\lceil 2^n t\rceil}(\omega)$$

For every $A\in\mathcal{B}(\mathbb{R})$,

$$\{(\omega,t)\in\Omega\times[0,T] : X_t^n(\omega)\in A\}$$
$$= \bigcup_{\substack{k:\\ 2^{-n}k<T}} \underbrace{\{X_{2^{-n}k}\in A\}}_{\in\mathcal{F}_T} \times \underbrace{[2^{-n}(k-1), 2^{-n}k)}_{\in\mathcal{B}([0,T])}.$$

Thus X^n is $\mathcal{F}_T\otimes\mathcal{B}([0,T])$-measurable. Take limits.

Sometimes it is also convenient to assume that the $\mathcal{F}_t$ contain all null sets, i.e. is complete.

<u>Defn</u>. A filtration satisfies the <u>usual conditions</u> if it is right-continuous and complete (w.r.t. a given probability measure).

1.2. Functions of bounded variation

Defn. The total variation of $x:[a,b]\to\mathbb{R}$ is

$$V_x[a,b] = \sup\Big\{ \sum_{i=1}^{n} |x(t_i)-x(t_{i-1})| : \underbrace{a\le t_0<\dots<t_n\le b}_{\text{partition of } [a,b]} \Big\}.$$

The space $BV[a,b]$ of functions of bounded variation consists of $x:[a,b]\to\mathbb{R}$ with $V_x[a,b]<\infty$.

Recall: A signed measure μ on $[a,b]$ is the difference $\mu=\mu_+-\mu_-$ (Hahn-Jordan decomposition) of two mutually singular finite positive measures μ_+ and μ_-. The total variation measure of μ is

$$|\mu| = \mu_+ + \mu_-$$

Hahn-Jordan thm: Given any finite positive measures μ_1 and μ_2 there are μ_+ and μ_- s.t. $\mu=\mu_1-\mu_2=\mu_+-\mu_-$.

Prop. (i) Given a signed measure μ on $[a,b]$, its CDF $x(t)=\mu([a,t])$ is in $BV[a,b]$, càdlàg, and

$$V_x[a,b] = |\mu|((a,b]).$$

(ii) Given $x\in BV[a,b]$ that is right-continuous, there is μ s.t. $x(t)=\mu([a,b])$.

Proof. (i) Let $\mu = \mu_+ - \mu_-$ be a signed measure and x its CDF. Let $a \le t_0 < t_1 < \cdots < t_n \le b$. Then

$$\sum_{i=1}^{n} |x(t_i) - x(t_{i-1})| = \sum_{i=1}^{n} |\mu(t_{i-1}, t_i]| \le |\mu|(a,b]$$

$$\Rightarrow V_x[a,b] \le |\mu|(a,b].$$

For the other direction, let $(t_i^{(m)})_{i=0}^{n_m}$ be a sequence of nested partitions with $\max_i |t_i^{(m)} - t_{i-1}^{(m)}| \xrightarrow{m\to\infty} 0$. It suffices to show that

$$|\mu|(a,b] = \lim_{m\to\infty} \sum_{i=1}^{n_m} |x(t_i^{(m)}) - x(t_{i-1}^{(m)})|.$$

To see this, define the probability on $(a,b]$ by

$$\nu(dt) = \frac{|\mu|(dt)}{|\mu|(a,b]}.$$

Let $\mathcal{F}_m = \sigma((t_{i-1}^{(m)}, t_i^{(m)}] : 1 \le i \le n_m) \subset \mathcal{F}_{m+1}$

$$X = \frac{d\mu}{d|\mu|} = 1_{\operatorname{supp}\mu_+} - 1_{\operatorname{supp}\mu_-}$$

$$X_m = E[X \mid \mathcal{F}_m].$$

For $s \in (t_{i-1}^{(m)}, t_i^{(m)}]$ then

$$X_m(s) = \frac{\mu((t_{i-1}^{(m)}, t_i^{(m)}])}{|\mu|((t_{i-1}^{(m)}, t_i^{(m)}])} = \frac{x(t_i^{(m)}) - x(t_{i-1}^{(m)})}{|\mu|((t_{i-1}^{(m)}, t_i^{(m)}])}$$

$$\Rightarrow E|X_m| = \frac{1}{\mu((a,b])} \sum_{i=1}^{n_m} |x(t_i^{(m)}) - x(t_{i-1}^{(m)})|$$

and the claim is $E|X_m| \to 1$. But (X_m) is a bd martingale, so there is Y s.t. $X_m \to Y$ in L^1

$\Rightarrow E|X_m| \to E|Y|$.

Since $\sigma(\bigcup \mathcal{F}_n) = \mathcal{B}((a,b])$ in fact $X = Y$ a.s., so

$$E|Y| = E|X| = 1.$$

(ii) Now let $x \in BV[a,b]$. Assume $x(a) = 0$ and set

$$x_\pm(t) = \tfrac{1}{2}(V_x[a,t] \pm x(t)).$$

Claim: $x_\pm$ are increasing, i.e. $x_\pm(s) \geq x_\pm(t)$ if $s > t$.

Let $a \leq t_0 < \cdots < t_n \leq t$ be a partition of $[0,t]$, s.t. $a \leq t_0 < \cdots < t_n \leq t \leq s$ is a partion of $[0,s]$.

$$\Rightarrow 2x_\pm(s) = V_x[a,s] \pm x(s)$$

$$\geq \underbrace{\sum |x(t_i) - x(t_{i-1})|}_{\geq V_x[a,t] - \varepsilon} + \underbrace{|x(s) - x(t)| \pm x(s)}_{\geq \pm x(t)}$$

$$\Rightarrow x_\pm(s) \geq x_\pm(t).$$

Claim: If x is right-continuous, so are $x_\pm$.

It suffices to show $v(t) = V_x[a,t]$ is right-continuous.
Let $\beta = v(t+) - v(t)$. Need to show $\beta = 0$. Let $h>0$ s.t.

$$v(t+h) - v(t) < \beta + \varepsilon$$
$$|x(s) - x(t)| < \varepsilon \quad \text{for } s \in [t, t+h].$$

There is a partition of $[t, t+h]$ s.t.

$$\sum_{i=1}^{n} |x(t_i) - x(t_{i-1})| \geq \frac{3}{4} V_x[t, t+h] \geq \frac{3}{4}\beta$$
$$\Rightarrow \quad \sum_{i=2}^{n} |x(t_i) - x(t_{i-1})| \geq \frac{3}{4}\beta - |x(t_1) - x(t_0)| \geq \frac{3}{4}\beta - \varepsilon$$

Since $\beta \leq v(t_1) - v(t_0)$ there is a part of $[t_0, t_1]$ s.t.

$$\sum |x(t_i') - x(t_{i-1}')| \geq \frac{3}{4}\beta$$

Thus there is a partition of $[t, t+h]$ s.t.

$$V_x[t, t+h] \geq \frac{3}{2}\beta - \varepsilon$$

On the other hand,

$$V_x[t, t+h] \leq \beta + \varepsilon$$

$$\Rightarrow \beta \leq 4\varepsilon \ \forall \varepsilon > 0 \Rightarrow \beta = 0.$$

<u>Defn</u>. Let $x \in BV[a,b]$ be right-continuous with associated signed measure μ. For $f \in L^1(|\mu|)$ the Lebesgue-Stieltjes integral is defined by

$$\int_s^t f(u)\,dx(u) = \int_{(s,t]} f(u)\,\mu(du) \qquad (a \le s < t \le b)$$

$$\int_s^t f(u)\,|dx(u)| = \int_{(s,t]} f(u)\,|\mu|(du)$$

and set $f \circ x(t) = \int_0^t f(u)\,dx(u)$.

<u>Exercise</u>. Let $x \in BV[a,b]$ be right-continuous. Then

$$\left|\int_a^b f(t)\,dx(t)\right| \le \int_a^b |f(t)|\,|dx(t)|$$

and $f \circ x$ is in BV and right-cont. with signed measure $f(t)\,dx(t)$ and TV $|f(t)|\,|dx(t)|$.

<u>Prop</u>. Let $x \in BV$ be right-cont. and f bounded and left-cont. Then for any sequence of partitions $(t_i^{(m)})_{i=1}^{n_m}$ of $[a,b]$ with step size $\xrightarrow{m\to\infty} 0$:

$$\int_a^b f(t)\,dx(t) = \lim_{m\to\infty} \sum_{i=1}^{n_m} f(t_{i-1}^{(m)})\left(x(t_i) - x(t_{i-1})\right)$$

$$\int_0^b f(t)\,|dx(t)| = \lim_{m\to\infty} \sum_{i=1}^{n_m} f(t_{i-1}^{(m)})\,|x(t_i) - x(t_{i-1})|.$$

Proof. Let $f_m(a)=0$, $f_m(t)=f(t_{i-1}^{(m)})$ if $t\in(t_{i-1}^{(m)},t_i^{(m)}]$.

$\Rightarrow f(t)=\lim_{m\to\infty} f_m(t)$ by left-continuity.

$$\Rightarrow \sum_{i=1}^{n_m} f(t_{i-1}^{(m)})\left(x(t_i^{(m)})-x(t_{i-1}^{(m)})\right)=\int_{(a,b]} f_m(t)\,dx(t)$$

$$\overset{DCT}{\to} \int f(t)\,dx(t).$$

The second claim is similar.

__Exercise__. Let $x,y\in BV[a,b]$ be right-cont. Then

$$x(b)y(b)-x(a)y(a)=\int_a^b x(t)\,dy(t)+\int_a^b y(t-)\,dx(t)$$

1.3. Continuous-time martingales

Defn. Given a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$, a martingale is a family of random variables M_t, $t \in \mathbb{T}$ such that

(i) $(t \mapsto M_t) \in D(\mathbb{T})$, P-a.s. [if $\mathbb{T}$ is continuous]

(ii) $\forall t \geq 0$: $M_t \in \mathcal{F}_t$ and $E|M_t| < \infty$.

(iii) $\forall\, 0 \leq s \leq t$: $E[M_t \mid \mathcal{F}_s] = M_s$, P-a.s.

If $(t_k) \subset \mathbb{T}$ then $(M(t_k))_k$ is a discrete-time martingale w.r.t. $(\mathcal{F}_{t_k})$. By taking $t_k = 2^{-n}k$ and using right-continuity, most properties extend from discrete to continuous.

Doob's inequality. If (M_t) is a martingale and

$$E_\ell = \{\omega : \sup_{0 \leq s \leq t} |M_s(\omega)| \geq \ell\}$$

then

$$P[E_\ell] \leq \frac{1}{\ell} E[1_{E_\ell} |M_t|] \leq \frac{1}{\ell} E[|M_t|]$$

$$P[E_\ell] \leq \frac{1}{\ell^2} E[1_{E_\ell} |M_t|^2] \leq \frac{1}{\ell^2} E[|M_t|^2].$$

Proof. The discrete-time versions of Doob's inequalities imply the statement with E_ℓ replaced by

$$E_\ell^n = \{ \sup_{\substack{0 \le s \le t \\ s \in 2^{-n} t \mathbb{N}}} |M_s| \ge \ell \}.$$

By right-continuity, $P(E_\ell^n) \uparrow P(E_\ell)$.

Defn. (X_t) is a submartingale if (i) & (ii) hold,

(iii') $\forall\, 0 \le s \le t$: $E[X_t | \mathcal{F}_s] \ge X_s$, P-a.s.

and a supermartingale if $E[X_t | \mathcal{F}_s] \le X_s$.

Exercise. Let (X_t) be a martingale, $\phi: \mathbb{R} \to \mathbb{R}$ convex, and assume $E[|\phi(X_T)|] < \infty$. Then $\phi(X_t)$ is integrable for any $t \le T$ and a submartingale.

Defn. A stopping time is a random variable τ with values in $[0, \infty]$ s.t.

$\forall t \ge 0$: $E_t = \{ \tau \le t \} \in \mathcal{F}_t$.

Define $\mathcal{F}_\tau = \{ A \in \mathcal{F} : A \cap E_t \in \mathcal{F}_t \}$.

<u>Exercise</u>. • If τ_1, τ_2 are stopping times, so are $\tau_1 \wedge \tau_2$ and $\tau_1 \vee \tau_2$.

- If $\tau_1 \le \tau_2$ are stopping times, $\mathcal{F}_{\tau_1} \subseteq \mathcal{F}_{\tau_2}$.
- If f is a measurable function s.t. $f(t) \ge t$ for all t then $f(\tau)$ is a stopping time if τ is one. Thus $\tau_n = 2^{-n} \lceil 2^n \tau \rceil$ is a stopping time s.t. $\tau_n \ge \tau$ and $\tau_n \to \tau$.

<u>Exercise</u>: Let X be a random variable in L^1, i.e. $E|X| < \infty$. Then the set of random variables $E[X \mid \mathcal{G}]$ where $\mathcal{G} \subset \mathcal{F}$ ranges over sub-σ-fields is uniformly integrable: $\forall \varepsilon > 0 \; \exists \lambda > 0$ s.t. $\forall \mathcal{G}$:

$$E\left[|E[X|\mathcal{G}]| \, 1_{|E[X|\mathcal{G}]| > \lambda} \right] \le \varepsilon .$$

<u>Vitali's convergence theorem</u>: Equivalent:

- $X_j \in L^1$ and $X_j \to X$ in L^1
- X_j UI and $X_j \to X$ in probability.

Thm (OST). Let (M_t) be a martingale and $\tau_1 \le \tau_2$ be two bounded stopping times. Then $M_{\tau_i} \in L^1$ and

$$E[M_{\tau_2} | \mathcal{F}_{\tau_1}] = M_{\tau_1} \quad \text{a.s.}$$

Proof. Assume $\tau_1, \tau_2 \le N$. Let $\tau_i^n = 2^{-n}\lceil 2^n \tau_i \rceil \le N+1$ where N is a deterministic constant. The discrete-time OST implies that

$$M_{\tau_2^n} = E[M_{N+1} | \mathcal{F}_{\tau_2^n}]$$

In particular, $M_{\tau_2^n}$ is UI. Since $M_{\tau_2^n} \to M_{\tau_2}$ a.s. therefore $M_{\tau_2^n} \to M_{\tau_2}$ in L^1 and likewise for τ_1. The discrete OST also implies

$$E[M_{\tau_2^n} | \mathcal{F}_{\tau_1^n}] = M_{\tau_1^n}$$

In particular, for any $A \in \mathcal{F}_{\tau_1} \subseteq \mathcal{F}_{\tau_1^n}$,

$$E[M_{\tau_2^n} 1_A] = E[M_{\tau_1^n} 1_A]$$

Since $M_{\tau_i^n} \to M_{\tau_i}$ in L^1 the claim follows.

Cor. For any stopping time, $M_{t\wedge\tau}$ is a martingale.

Exercise. Extend to (M_t) a submartingale.

1.4. Semimartingales

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ be a filtered probability space.

<u>Example</u> Let (X_n) be an adapted discrete-time process. Define

$$A_n = \sum_{j=1}^{n} E[X_j - X_{j-1} \mid \mathcal{F}_{j-1}], \qquad A_0 = 0.$$

Then $Y_n = X_n - A_n$ is a martingale: for $m \leq n$,

$$E[Y_n \mid \mathcal{F}_m] = E[X_n \mid \mathcal{F}_m] - \sum_{j=1}^{n} \underbrace{E[E[X_j - X_{j-1} \mid \mathcal{F}_{j-1}] \mid \mathcal{F}_m]}_{= \begin{cases} E[X_j - X_{j-1} \mid \mathcal{F}_m] & (j > m) \\ E[X_j - X_{j-1} \mid \mathcal{F}_{j-1}] & (j \leq m) \end{cases}}$$

$$= X_m - \sum_{j=1}^{m} E[X_j - X_{j-1} \mid \mathcal{F}_{j-1}].$$

and $A_n \in \mathcal{F}_{n-1}$. The decomposition $X_n = Y_n + A_n$ with these properties is the Doob or <u>semimartingale decomposition</u> of X_n. The process Y_n is the martingale part of X_n and A_n the compensator.

Exercise: The semimartingale decomposition is unique.

The semimartingale decomposition is more subtle in the continuous case.

Prop. Let (A_t) be a continuous-time martingale. If A is continuous and BV, P-a.s., then A is const.

Proof. Assume $A_0=0$ and let A^*_t be the variation of A on $[0,t]$. It suffices to consider $A^*(t)\le C$. Otherwise, consider $A_{t\wedge\tau}$ where $\tau=\inf\{t: A^*_t\ge C\}$.

Let $t_j=(j/n)T$. Then

$$\sum_{j=1}^n |A_{t_j}-A_{t_{j-1}}|^2 \le \sup_j |A_{t_j}-A_{t_{j-1}}| \times \sum_j |A_{t_j}-A_{t_{j-1}}|$$
$$\le C\sup_j |A_{t_j}-A_{t_{j-1}}|.$$

Since $|A_{t_j}|\le |A_0|+C$ and A is continuous on $[0,T]$, thus uniformly continuous,

$$E\left[\sum_{j=1}^n |A_{t_j}-A_{t_{j-1}}|^2\right]\to 0.$$

On the other hand, since A is a martingale,

$$E\left[\sum_{j=1}^n |A_{t_j}-A_{t_{j-1}}|^2\right] = E\left[|A_T-A_0|^2\right]$$

Thus $E\left[|A_T-A_0|^2\right]=0$, i.e. $A_T=A_0$ a.s.

Prop. Let (M_t) be a bounded martingale and (A_t) be an adapted continuous process of finite variation with $A_t^* \le C_t$ for every t. Then

$$X_t = M_t A_t - M_0 A_0 - \int_0^t M_s \, dA_s$$

is a martingale.

Rk. Formally, $X_t = \int A_s \, dM_s$.

Proof. Since A is continuous,

$$\int_0^t M_s \, dA_s = \lim_{n\to\infty} \sum_{j=1}^n M_{t_j} (A_{t_j} - A_{t_{j-1}})$$

$$= M_t A_t - M_0 A_0 + \lim_{n\to\infty} \sum_{j=1}^n A_{t_{j-1}} (M_{t_j} - M_{t_{j-1}})$$

$$\Rightarrow X_t = \lim_{n\to\infty} \sum_{j=1}^n A_{t_{j-1}} (M_{t_j} - M_{t_{j-1}}) = \lim_{n\to\infty} X_t^n$$

Since $A_{t_{j-1}} \in \mathcal{F}_{t_{j-1}}$ and M is a martingale,

$$E\left[A_{t_{j-1}} (M_{t_j} - M_{t_{j-1}}) \mid \mathcal{F}_{t_{j-1}} \right] = 0.$$

and similarly X^n is a martingale.

If M is bd, $X_t^n \to X_t$ in L^1. It follows that X is a martingale.

2. Brownian motion

In fact, suffices to have t in dense set.

2.1. Kolmogorov's continuity criterion

<u>Thm.</u> (Kolmogorov's criterion). Assume $(X_t)_{t\in\mathbb{R}^d}$ is a stochastic process s.t. for some $p\geq 1$, $\alpha>0$,

$$\mathbb{E}\left[|X_t-X_s|^p\right]\leq A\,|t-s|^{d+\alpha}, \qquad 0\leq t,s\leq 1.$$

Then there is a version $\tilde{X}$ of X, i.e., $P(X_t=\tilde{X}_t)=1\ \forall t$, that is continuous, in fact γ-Hölder continuous:

$$P\left[\sup_{t\neq s}\frac{|\tilde{X}_t-\tilde{X}_s|}{|t-s|^\gamma}\geq\lambda\right]\leq C\,\frac{A}{\lambda^\beta}$$

provided that $\gamma<\alpha/p$ and with $C=C_{\alpha,\gamma,p}$.

The essence is really a real analysis estimate. Formally, the estimate follows taking expectation over the following inequality ("Besov space embedding").

<u>Prop.</u> For $x:\mathbb{R}^d\to\mathbb{R}^n$ <u>continuous</u>, $p\geq 1$ and $\gamma>0$,

$$\underbrace{\sup_{\substack{t\neq s\\ t,s\in B}}\frac{|x(t)-x(s)|}{|t-s|^\gamma}}_{[x]_\gamma}\leq C_{\gamma,p}\underbrace{\left(\iint_{B\,B}\frac{|x(t)-x(s)|^p}{|t-s|^{\gamma p+2d}}\,dt\,ds\right)^{1/p}}_{[\![x]\!]_{\gamma+2d/p,\,p}} \qquad (*)$$

where $B=B_1(0)\subset\mathbb{R}^d$.

<u>Garsia-Rodemich-Rumsey inequality</u>. Let p and Ψ be strictly increasing continuous functions on $(0,\infty)$ s.t. $p(0)=0$ and $\Psi(t)\to\infty$ $(t\to\infty)$. Then if $x:[0,T]\to\mathbb{R}^n$ is <u>continuous</u> and

$$\int_0^T\int_0^T \Psi\left(\frac{|x(t)-x(s)|}{p(|t-s|)}\right) ds\, dt \le B < \infty$$

then

$$|x(t)-x(s)| \le 8\int_0^{|t-s|} \Psi^{-1}\left(\frac{4B}{u^2}\right) p(du) \quad p'(u)\,du$$

<u>Proof of (*) when $d=1$</u>. Let $\Psi(u)=u^p$, $p(u)=u^{\gamma+2/p}$,

$$B=B(x)=\int_0^1\int_0^1 \frac{|x(t)-x(s)|^p}{|t-s|^{\gamma p+2}}\, ds\, dt.$$

$$\Rightarrow |x(t)-x(s)| \le C_p\int_0^{|t-s|}\left(\frac{4B}{u^2}\right)^{1/p} u^{\gamma+2/p-1}\,du$$

$$= C_p' B^{1/p}\int_0^{|t-s|} u^{\gamma-1}\,du \le C_{\gamma,p} B^{1/p}|t-s|^\gamma$$

<u>Rk</u>. If x is only in L^1_{loc} and the RHS is finite, there is $\tilde{x}$ s.t. $\tilde{x}=x$ a.e. s.t. the estimate holds: Set

$$x^\varepsilon(t)=\frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} x(t+u)\,du$$

Then $[x^\varepsilon]_\gamma$ is bounded. By Arzela-Ascoli there is $\tilde{x}$ s.t. $\|x^\varepsilon-\tilde{x}\|_\infty\to 0$ along some $\varepsilon\downarrow 0$. But $x^\varepsilon\to x$ a.e. by Lebesgue diff. thm., so $\tilde{x}=x$ a.e.

Proof of Kolmogorov's criterion. If we already knew that X was continuous, the estimate would follow from the GRR inequality. But we have not even assumed that $t \mapsto X_t$ is Borel measurable!

Let

$$X_t^n = X_{2^{-n}\lfloor 2^n t \rfloor}$$

Then X_t^n is measurable. Set

$$\hat{X}_t(\omega) = \begin{cases} \lim\limits_{n\to\infty} X_t^n(\omega) & \text{for } (\omega,t) \text{ s.t. the limit exists} \\ 0 & \text{else.} \end{cases}$$

Since $P[|X_t^n - X_t| \geq 2^{-n}] \leq A\, 2^{+n}(2^{-n})^{1+\beta}$ is summable the limit exist a.s. for every fixed t.

Let

$$Y = [\![\hat{X}]\!] = \left(\int_0^1 \int_0^1 \frac{|\hat{X}_t - \hat{X}_s|^p}{|t-s|^{\gamma p+2}} \, ds\, dt \right)^{1/p}$$

$$\Rightarrow E[Y^p] = A \int_0^1\int_0^1 |t-s|^{1+\alpha-p\gamma-2} \, ds\, dt < \infty \text{ if } \alpha > \gamma p.$$

Thus, for every ω, $\hat{X}$ coincides for a.e. t with a $\tilde{X}$ s.t.

$$[\tilde{X}]_\gamma \leq CY \qquad \text{for all } \omega.$$

It remains to see that $P[X_t = \tilde{X}_t] = 1$ for every t.
It suffices to show this for $t = 2^{-n}k$. Then

$$P[X_t = \hat{X}_t] = P[\hat{X}_t = \tilde{X}_t].$$

But $P[|\hat{X}_t - \hat{X}_t^\varepsilon| > \delta] \lesssim \delta^{-1}\varepsilon^{1+\beta} \to 0$, so as before $\hat{X}_t^\varepsilon \to \hat{X}_t$ a.s. as $\varepsilon \to 0$, so $\hat{X}_t = \tilde{X}_t$ a.s.

<u>Proof</u> (GRR). Let

$$I(t) = \int_0^T \Psi\left(\frac{|x(t) - x(s)|}{p(|t-s|)} \right) ds$$

so that $\int_0^T I(t)\,dt = B$. By the mean-value theorem, there is $t_0 \in (0,T)$ s.t. $I(t_0) \leq B/T$.

Given t_{n-1}, define d_{n-1} by

$$p(d_{n-1}) = \tfrac{1}{2} p(t_{n-1}) \qquad (\text{so } 0 < d_{n-1} < t_{n-1})$$

and $t_n \in (0, d_{n-1})$ s.t.

$$I(t_n) \leq \frac{2B}{d_{n-1}} \quad \& \quad \Psi\left(\frac{|x(t_n) - x(t_{n-1})|}{p(|t_n - t_{n-1}|)} \right) \leq 2\,\frac{I(t_{n-1})}{d_{n-1}}.$$

Each can fail on set of measure $< \frac{d_{n-1}}{2}$ of t_n.
Thus both can fail only on a set of meas. $< d_{n-1}$.

Thus

$$t_0 > d_0 > t_1 > d_1 > \cdots$$

Then:

$$p(d_{n+1}) = \tfrac{1}{2} p(t_{n+1}) \le \tfrac{1}{2} p(d_n) \Rightarrow d_n, t_n \downarrow 0.$$

Also:

$$p(t_n - t_{n+1}) \le p(t_n) = 2p(d_n) = 4\left(p(d_n) - \tfrac{1}{2}p(d_n)\right)$$
$$\le 4\left(p(d_n) - p(d_{n+1})\right)$$

$$\Rightarrow |x(t_{n+1}) - x(t_n)| \le p(t_n - t_{n+1})\, \Psi^{-1}\left(\frac{2I(t_n)}{d_n}\right)$$
$$\le 4\left(p(d_n) - p(d_{n+1})\right) \Psi^{-1}\Big(\underbrace{\frac{4B}{d_n d_{n-1}}}_{\le \frac{4B}{d_n^2}}\Big)$$
$$\le 4 \int_{d_{n+1}}^{d_n} \Psi^{-1}\left(\frac{4B}{u^2}\right) p(du)$$

$$\Rightarrow |x(t_0) - x(0)| \le 4 \int_0^T \Psi^{-1}\left(\frac{4B}{u^2}\right) p(du)$$

By an analogous argument,

$$|x(T) - x(t_0)| \le 4 \int_0^T \Psi^{-1}\left(\frac{4B}{u^2}\right) p(du)$$

$$\Rightarrow |x(T) - x(0)| \le 8 \int_0^T \Psi^{-1}\left(\frac{4B}{u^2}\right) p(du).$$

Given $0 \le s < t \le T$ set

$$\bar{x}(u) = x\left(s + \frac{(t-s)}{T} u\right)$$

$$\bar{p}(u) = p\left(\frac{(t-s)}{T} u\right)$$

The argument above with $B \rightsquigarrow \left(\frac{T}{t-s}\right)^2 B = \bar{B}$ gives

$$|x(t) - x(s)| \le 8 \int_0^T \Psi^{-1}\left(\frac{4\bar{B}}{u^2}\right) \bar{p}(du)$$

$$= 8 \int_0^{t-s} \Psi^{-1}\left(\frac{4B}{u^2}\right) p(du).$$

2.2. Definition of Brownian motion

Defn. A stochastic process $(B_t)_{t\geq 0}$ is called a Brownian motion or Wiener process if it is continuous and:

(i) $\forall t<s: \; B_t - B_s \sim \mathcal{N}(0, t-s)$.

(ii) $\forall t<s: \; B_t - B_s$ is independent of $\sigma(B_u : u \leq s)$.

Thm. There exists a unique probability measure on $C[0,\infty)$ such that the canonical process is a Wiener process. This is the Wiener measure.

Lemma. If (i) holds for a stochastic process on some probability space then there is a version that is in fact Hölder continuous for any Hölder exponent strictly less than $1/2$.

Proof. Since $W_t - W_s \sim \mathcal{N}(0, t-s)$,

$$E\left[|W_t - W_s|^p\right] \leq C_p |t-s|^{p/2} \text{ for any } p \geq 1.$$

Thus Kolmogorov applies $\gamma = \frac{1}{2} - \varepsilon$ when $p \geq \frac{1}{\varepsilon}$:

$$E\left[|W_t - W_s|^p\right] \leq C_p |t-s|^{p/2} \leq C_p |t-s|^{\gamma p + 1}$$

Proof (construction of Wiener measure). Given any probability space $(\Omega, \mathcal{F}, P)$ on which a Brownian motion (B_t) is defined, consider the map

$$I: \Omega \longrightarrow C[0,\infty), \quad \omega \mapsto (t \mapsto B_t(\omega)).$$

This map is measurable. Thus $P \circ I^{-1}$ defines a Borel measure on $C[0,\infty)$.

Using a monotone class argument one checks it is uniquely defined by its fin-dim. distributions.

Prop. Brownian motion is not in $C^{1/2}$:

$$P\left[\sup_{\substack{t \neq s \\ 0<t,s<1}} \frac{|X_t - X_s|}{|t-s|^{1/2}} = \infty\right] = 1.$$

Proof. For each $t>s$, $(B_t - B_s)/|t-s|^{1/2}$ is $\sim \mathcal{N}(0,1)$.

$$\sup_{t \neq s} \frac{|X_t - X_s|}{|t-s|^{1/2}} \geq \sup_{0 \leq k < 2^n} \frac{|X_{2^{-n}(k+1)} - X_{2^{-n}k}|}{2^{-n/2}}$$

$$\Rightarrow P\left[\sup \frac{|X_t - X_s|}{|t-s|^{1/2}} > \lambda\right] \leq 1 - P[Z < \lambda]^{2^n} \to 0.$$

where $Z \sim \mathcal{N}(0,1)$.

There are many ways of constructing (B_t).

<u>Defn</u>. Let $(\Omega, \mathcal{F}, P)$ be a probability space. Then $S \subset L^2(\Omega, \mathcal{F}, P)$ is a <u>Gaussian space</u> if S is a closed linear subspace and any X is a Gaussian random variable.

<u>Example</u>. Let (X_i) be i.i.d $\mathcal{N}(0,1)$ on some probability space. Then $\overline{\mathrm{span}\{X_i\}}$ is a Gaussian space. The X_i are an orthonormal system:

$$E[X_i X_j] = \delta_{ij}$$

Note that limits of Gaussian random variables are Gaussian.

<u>Prop</u>. Let H be a separable Hilbert space and S as in the example. Then there is an isometry $I: H \to S$. Thus:

- $\forall f \in H: \; I(f) \sim \mathcal{N}(0, \|f\|_H^2)$
- $\forall f, g \in H: \; E\, I(f) I(g) = (f,g)_H$

In fact, $I(\alpha f + \beta g) = \alpha I(f) + \beta I(g)$ a.s.

Proof. Let (e_i) be an orthonormal basis for H. Set

$$I(f) = \sum_{i=1}^{\infty} (f, e_i) X_i .$$

The limit exists in L^2 and a.s. since

$$k \mapsto \sum_{i=1}^{k} (f, e_i) X_i$$

is a martingale bounded in L^2.

Defn. Let $\dot{W}$ be an isometry from $L^2(\mathbb{R}_+)$ into some Gaussian space. Then $\dot{W}$ is called *white noise*. For $t \geq 0$, define

$$W_t = \dot{W}(1_{[0,t]})$$

Rk. Would like to think of $\dot{W}(f)$ as $\int f \, dW$ but W has infinite variation almost surely.

Exercise. (i) For $A \subset \mathbb{R}_+$ Borel, $|A| < \infty$, $\dot{W}(A) \sim N(0, |A|)$

(ii) For $A, B \subset \mathbb{R}_+$ with $A \cap B = \emptyset$, $\dot{W}(A)$ and $\dot{W}(B)$ are independent.

(iii) For $A = \bigcup_{i=1}^{\infty} A_i$ with A_i disjoint,

$$\dot{W}(A) = \sum_{i=1}^{\infty} \dot{W}(A_i) \text{ in } L^2 \text{ and a.s.}$$

$\dot{W}$ looks like a random measure but it is not.

Prop. For any $t_1, \dots, t_n$ the vector $(W_{t_i})_{i=1}^n$ is jointly Gaussian with covariance
$$\mathbb{E}[W_t W_s] = s \wedge t \quad \text{for } s,t \geq 0.$$
Moreover, $W_0 = 0$ a.s. and
$$W_t - W_s \text{ is independent of } \sigma(W_u : u < s)$$
$$W_t - W_s \sim \mathcal{N}(0, t-s).$$

Exercise: Let B be a Brownian motion. Then

- $-B$ is a Brownian motion (symmetry)
- $\frac{1}{\lambda} B_{\lambda^2 t}$ is a Brownian motion (scale invariance)
- $B_{t+s} - B_s$ is a Brownian motion that is indep. of $(B_u : u \leq s)$ (Markov prop.).

Exercise. Understand how Brownian motion is the Gaussian measure with covariance $(-\Delta)^{-1}$ on $[0,\infty)$ with 0-boundary condition at 0.

Defn. A stochastic process taking values in $\mathbb{R}^d$ is a d-dim. Brownian motion if each component is a Brownian motion.

2.3. Heat equation and Feynman-Kac formula

Let $p_t(x,y) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{|x-y|^2}{2t}}$ be the heat kernel on R.

Defn. The semigroup of Brownian motion is

$$T_t f(x) = E_x[f(B_t)]$$

where E_x is the expectation of a BM starting at $x \in \mathbb{R}$.

Fact. Since $B_t - x \sim \mathcal{N}(0,t)$,

$$T_t f(x) = \int p_t(x,y) f(y) = \int f(x+y) p_t(0,y)\, dy$$
$$= \int f(x+y\sqrt{t})\, p_1(0,y)\, dy.$$

Defn. The infinitesimal generator is

$$Lf(x) = \lim_{t \downarrow 0} \frac{T_t f(x) - f(x)}{t}$$

for f such that the limit exists.

Fact. For $f \in C^2$ the limit exists and

$$Lf = \frac{1}{2} f''.$$

Proof. If $f \in C^3$,

$$\lim_{t \to 0} \int \frac{f(x+y\sqrt{t})-f(x)}{t} \, p_1(0,y)\, dy$$

$$= \lim_{t \to 0} \int \frac{f'(x)y\sqrt{t} + \frac{1}{2}f''(x)y^2 t + t^{3/2}e_t(y)}{t} \, p_1(0,y)\, dy$$

with $|e_t(y)| \leq C|y|^2$. Thus

$$Lf(x) = \frac{1}{2} f''(x).$$

If $f \in C^2$, one still has $t^{3/2}|e_t(y)| = o(t)$.

Thus $u(t,x) = T_t f(x)$ satisfies the heat equation

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2}, \qquad u(t,x) \to f(x) \quad (t \to 0).$$

Exercise. Let $X_t = x + \sigma B_t + mt$ be Brownian motion with variance σ^2 and drift m. Show

$$Lf(x) = \frac{\sigma^2}{2} f''(x) + m f'(x).$$

Fact. Given $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$, a continuous adapted process (B_t) is a Brownian motion iff

$$E[f(B_t) \mid \mathcal{F}_s] = T_{t-s} f(B_s) \quad \forall f \text{ continuous}.$$

Feynman-Kac formula. Let V be bounded and continuous. Then

$$u(t,x) = E_x\left[\exp\left(\int_0^t V(B_s)\,ds\right) f(X_t)\right]$$

satisfies

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} + Vu, \qquad u(0,x) = f(x).$$

Proof. Expand exponential:

$$u(t,x) = \sum_{n=0}^{\infty} \frac{1}{n!} E_x\left[\int_0^t \cdots \int_0^t V(B_{t_1}) \cdots V(B_{t_n})\, f(X_t)\, dt_1 \cdots dt_n\right]$$

$$= \sum_{n=0}^{\infty} E_x\left[\int_{0<t_1<\cdots<t_n<t} V(B_{t_1}) \cdots V(B_{t_n})\, f(X_t)\, dt_1 \cdots dt_n\right]$$

$$= \sum_{n=0}^{\infty} E_x\left[\int_{0<t_n<\cdots<t_1<t} V(B_{t-t_1}) \cdots V(B_{t-t_n})\, f(X_t)\, dt_1 \cdots dt_n\right]$$

$$= \sum_{n=0}^{\infty} \int_{0<t_n<\cdots<t_1<t} T_{t-t_1} V T_{t_1-t_2} \cdots V T_{t_{n-1}-t_n} V T_{t_n} f \, dt_1 \cdots dt_n \,\Big]$$

Differentiate term by term.

<u>Example</u>: Arcsine law.

Let $\xi_t = \frac{1}{t}\int_0^t 1_{B_s>0}\, ds$.

$\Rightarrow \quad E[\xi_t] = \frac{1}{2}$.

What is the distribution of ξ_t?

$$\xi_t = \int_0^1 1_{B_{ts}>0}\, ds \overset{(d)}{=} \int_0^1 1_{\sqrt{t}B_s>0}\, ds = \int_0^1 1_{B_s>0}\, ds$$

$\Rightarrow \xi_t \sim \xi_1$.

<u>Prop</u>. $P[\xi_1 \le x] = \int_0^x \frac{1}{\pi\sqrt{y(1-y)}}\, dy = \frac{2}{\pi}\arcsin(\sqrt{x})$.

<u>Proof</u>. Let $V(x) = 1_{x>0}$. This V is not cont. and a more careful argument would involve a limiting argument to apply Feynman-Kac. Anyway,

$$u(t,x) = E_x\left[\exp\left(-\sigma\int_0^t V(B_s)\, ds\right) ds\right]$$

satisfies

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} - \sigma V u, \quad u(0,x) = 1.$$

Let $g(\lambda,x) = \int_0^\infty e^{-\lambda t} u(t,x)\, dt$. Then

$$\lambda g + \sigma V g - \frac{1}{2}\frac{\partial^2 g}{\partial x^2} = 0.$$

Thus

$$(\lambda+\sigma)g - \tfrac{1}{2}g'' = 1 \qquad (x>0)$$

$$\lambda g - \tfrac{1}{2}g'' = 0 \qquad (x<0)$$

and g should be bounded at $\pm\infty$ and continuously differentiable at 0. The solution is

$$g(\lambda, x) = \frac{1}{\sqrt{\lambda}}\frac{1}{\sqrt{\lambda+\sigma}}.$$

Thus

$$E\left[\int_0^\infty \exp\left(-\lambda t - \sigma\int_0^t V(B_s)\,ds\right) dt\right]$$

$$= E\left[\int_0^\infty \exp\left(-\lambda t - \sigma t\,\xi_t\right) dt\right]$$

$$= E\left[\int_0^\infty \exp\left(-\lambda t - \sigma t\,\xi_1\right) dt\right]$$

$$= E\left[\frac{1}{\lambda+\sigma\xi_1}\right] = \frac{1}{\sqrt{\lambda}}\frac{1}{\sqrt{\lambda+\sigma}}.$$

Thus $E\left[\frac{1}{1+\sigma\xi_1}\right] = \frac{1}{\sqrt{1+\sigma}}$. Differentiating:

$$E\left[\xi_1^n\right] = \frac{1}{\pi}\int_0^1 x^{n-\frac{1}{2}}(1-x)^{-\frac{1}{2}}\,dx.$$

Since the support is bd. moments characterize the distribution.

2.4. Strong Markov and reflection property

Markov property. Let $t>s$. Then

$$\forall A: P[B_t \in A \mid \mathcal{F}_s] = p_{t-s}(B_s, A) = \frac{1}{\sqrt{2\pi(t-s)}} \int_A e^{-\frac{(y-B_s)^2}{2(t-s)}} dy$$

$\Leftrightarrow$

$$\forall f \text{ bd}: P[f(B_t) \mid \mathcal{F}_s] = P_{t-s} f(B_s) = \int p_{t-s}(B_s, y) f(y)\, dy.$$

Proof. Immediate from independent increments.

Strong Markov property. Let τ be a stopping time. Then $B_{\tau+t} - B_\tau$ is again a BM independent of $\mathcal{F}_\tau$ conditional on $\tau<\infty$.

Proof. Let $\tau_n = 2^{-n}\lceil 2^n \tau \rceil$ and $t_k = 2^{-n}k$.

Given $A \in \mathcal{F}_{\tau+s}$ set $A_k = A \cap \{\tau_n = t_k\} \in \mathcal{F}_{t_k+s}$. Then

$$\begin{aligned} E[f(B_{\tau_n+t}) 1_{A_k}] &= E[f(B_{t_k+t}) 1_{A_k}] \\ &= E[T_{t-s} f(B_{t_k+s}) 1_{A_k}] \\ &= E[T_{t-s} f(B_{\tau_n+s}) 1_{A_k}] \end{aligned}$$

$$\overset{\sum_k}{\Rightarrow} E[f(B_{\tau_n+t}) 1_{A\cap\{\tau<\infty\}}] = E[T_{t-s} f(B_{\tau_n+s}) 1_{A\cap\{\tau<\infty\}}]$$

For f bounded, can take $n\to\infty$:

$$E[f(B_{\tau+t}) 1_{A\cap\{\tau<\infty\}}] = E[T_{t-s} f(B_{\tau+s}) 1_{A\cap\{\tau<\infty\}}]$$

$$\Rightarrow E[f(B_{\tau+t}) \mid \mathcal{F}_{\tau+s}] = E[T_{t-s} f(B_{\tau+s})] \quad \text{a.s. on } \tau<\infty$$

<u>Rk</u>. Let $\theta_s : C[0,\infty) \to C[0,\infty)$, $\theta_s \omega(t) = \omega(t+s)$.
Then for fin. stopping times

$$P_x[\theta_\tau \omega \in A \mid \mathcal{F}_\tau] = P_{x(\tau)}[A], \quad P_x\text{-a.s.}$$

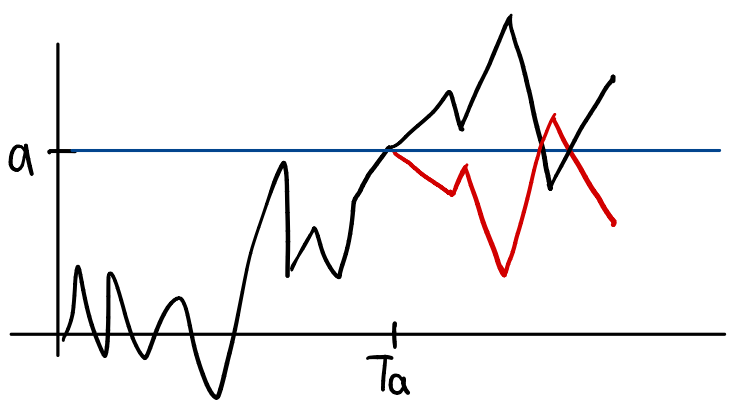
For a stopping time that is not a.s. finite,

$$P_x[\theta_\tau \omega \in A \mid \mathcal{F}_\tau] = P_{x(\tau)}[A], \quad P_x\text{-a.s. on } \tau<\infty.$$

<u>Reflection principle</u>. Let $a>0$, $T_a = \inf\{t \geq 0 : B_t = a\}$, and

$$X_t = \begin{cases} B_t & (t \leq T_a) \\ 2a - B_t & (t \geq T_a) \end{cases}$$

Then (X_t) is also a Brownian motion.



<u>Proof</u>. On the event $T_a < \infty$ set

$$\tilde{B}_t = B_{T_a + t} - B_{T_a}.$$

By the strong Markov property, $\tilde{B}$ is a BM cond. on $T_a < \infty$ indep. of $\mathcal{F}_{T_a}$. Same for $-\tilde{B}$.

$$\Rightarrow \quad B_t = B_{T_a \wedge t} + \tilde{B}_{(t-T_a)_+} 1_{T_a < \infty}$$

$$X_t = B_{T_a \wedge T} - \tilde{B}_{(t-T_a)_+} 1_{T_a < \infty}.$$

Thus B and X have the same distribution.

Cor. For $a \geq 0$ and $b \leq a$,

$$P\left[\sup_{0\leq s\leq t} B_s \geq a,\ B_t \leq b\right] = P\left[B_t \geq 2a-b\right]$$

In particular, $\sup_{0\leq s\leq t} B_s \sim |B_t|$ and

$$P\left[\sup_{0\leq s\leq t} |B_s| \geq a\right] \leq 4\, P\left[B_t \geq a\right]$$

Proof. For the first claim notice:

$$\text{LHS} = P\left[T_a \leq t,\ B_t \leq b\right]$$

$$\text{RHS} = P\left[2a - B_t \leq b\right]$$

$$= P\left[T_a \leq t,\ X_t \leq b\right] + \underbrace{P\left[T_a > t,\ B_t \geq 2a-b\right]}_{0}$$

For the second one:

$$P\left[\sup B_s \geq a\right] = \underbrace{P\left[\sup B_s \geq a,\ B_t \geq a\right]}_{P[B_t \geq a] \text{ trivially}}$$

$$+ \underbrace{P\left[\sup B_s \geq a,\ B_s \leq a\right]}_{P[B_t \geq a] \text{ by reflection}}$$

$$= 2\,P\left[B_t \geq a\right] = P\left[|B_t| \geq a\right]$$