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Problem 1. Find the unique strong solution to the SDE

$$dX_t = \frac{1}{2} X_t \, dt + \sqrt{1 + X_t^2} dB_t, \quad X_0 = x.$$

(Hint: consider the change of variables $Y_t = \sinh^{-1}(X_t)$.)

Problem 2. Show that $d_{\mathrm{TV}}(\mu, \nu) = \inf_{X\sim\mu, Y\sim\nu} P[X \neq Y]$ where the infimum is over all couplings of μ and ν.

Problem 3. Let τ_i be i.i.d. Exp(1) random variables. Define

$$N_t = \begin{cases} 0 & (0 \leq t < \tau_1) \\ k & (\tau_1 + \cdots + \tau_k \leq t < \tau_1 + \cdots + \tau_{k+1}). \end{cases}$$

Show that N_t is a Poisson process of rate $\lambda = 1$.

Problem 4. Let N be a Poisson process of rate $\lambda > 0$. Show that $N_t - \lambda t$ and $(N_t - \lambda t)^2 - \lambda t$ are martingales.

Problem 5. For a compound Poisson process with Levy measure M where we assume that M has two moments, show that $X_t - At$ and $(X_t - At)^2 - Bt$ where $A = \int x \, M(dx)$ and $B = \int x^2 \, M(dx)$ are martingales and that the generator of the process is

$$Lf(x) = \int (f(x+z) - f(x)) \, M(dz).$$

Problem 6. Let X be a Markov jump process with generator L. Let $\lambda > 0$ and show that

$$\begin{aligned} \lambda U - LU &= g \quad (x \in A) \\ U &= f \quad (x \notin A) \end{aligned}$$

can be uniquely solved and that the solution is given by

$$U(x) = E_x \left[e^{-\lambda \tau_A} f(X_{\tau_A}) + \int_0^{\tau_A} e^{-\lambda s} g(X_s) \, ds \right].$$