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Problem 1. Suppose that σ and b are Lipschitz functions. Explain why uniqueness in law holds for the SDE $dX_t = \sigma(X_t)dB_t + b(X_t)dt$.

Problem 2. Let b be bounded and σ be bounded and continuous functions from $\mathbb{R}$ to $\mathbb{R}$.

i. Suppose that X is a weak solution of the SDE $dX_t = b(X_t)dt + \sigma(X_t)dW_t$. Show that the process

$$f(X_t) - \int_0^t \left(b(X_s)f'(X_s) + \frac{1}{2}\sigma^2(X_s)f''(X_s) \right) ds$$

is a local martingale for all $f \in C^2$.

ii. Let X be a continuous, adapted process such that

$$f(X_t) - \int_0^t \left(b(X_s)f'(X_s) + \frac{1}{2}\sigma^2(X_s)f''(X_s) \right) ds$$

is a local martingale for each $f \in C^2$. Suppose that $\sigma(x) > 0$ for all x. Using Problem 3 of Homework 6, show that there exists a Brownian motion such that $dX_t = b(X_t)dt + \sigma(X_t)dW_t$.

Problem 3. Let W be a standard Brownian motion.

i. Let $B_t = W_t - tW_1$. Show that $(B_t)_{t\in[0,1]}$ is a continuous, mean-zero Gaussian process. What is the covariance $\mathbb{E}[B_s B_t]$?

ii. Is B adapted to the filtration generated by W?

iii. Let

$$dX_t = -\frac{X_t}{1-t}dt + dW_t, \quad X_0 = 0.$$

Verify that

$$X_t = (1-t)\int_0^t \frac{dW_s}{1-s} \quad \text{for} \quad 0 \le t < 1.$$

Show that $X_t \to 0$ as $t \uparrow 1$.

iv. Show that X is a continuous, mean-zero Gaussian process with the same covariance as B, which we call a *Brownian bridge*.

The following bonus problems are optional:

v. For $y \in \mathbb{R}$, define a process B^y by $B_t^y = B_t + ty = W_t + t(y - W_1)$, $0 \le t \le 1$. Let $F : C[0,1] \to \mathbb{R}$ be bounded, and continuous with respect to the uniform norm, and define

$$f(y) = \mathbb{E}\left[F(B^y)\right]. \tag{1}$$

Show that f is bounded and continuous, and that $\mathbb{E}(F(W)|W_1) = f(W_1)$ almost surely.

vi. For $\epsilon > 0$, let us write μ_ϵ for the probability measure on $C[0,1]$ given by

$$\mu_\epsilon(A) = \frac{\mathbb{P}(W \in A \text{ and } |W_1| < \epsilon)}{\mathbb{P}(|W_1| < \epsilon)} \tag{2}$$

and let μ_0 be the law of B. Use the previous part to show that $\mu_\epsilon \to \mu_0$ weakly as $\epsilon \downarrow 0$, so that B is the weak limit of a Brownian motion W conditioned on $\{|W_1| < \epsilon\}$. In this way, we say that "B is a Brownian motion conditioned on $B_1 = 0$", even though this is a 0-probability event.

Problem 4. Show that the SDE

$$dX_t = 3X_t^{1/3}dt + 3X_t^{2/3}dB_t, \quad X_0 = 0$$

has strong existence but not pathwise uniqueness.