Roland Bauerschmidt ⟨bauerschmidt@cims.nyu.edu⟩ *Spring 2026*

Problem 1. Suppose that M is a continuous local martingale with $\langle M\rangle_t \to \infty$ almost surely as $t \to \infty$. Show that $M_t/\langle M\rangle_t \to 0$ as $t \to \infty$ and conclude that $\mathcal{E}(M)_t \to 0$ almost surely.

Problem 2. Let B be a standard Brownian motion and, for $a, b > 0$, let $\tau_{a,b} = \inf\{t \geq 0 : B_t + bt = a\}$. Use Girsanov's theorem to prove that the density of $\tau_{a,b}$ is given by

$$a(2\pi t^3)^{-1/2}\exp(-(a-bt)^2/2t).$$

[You may use the result for $b = 0$ already proven in class.]

Problem 3. *i.* Let $f : \mathbb{C} \to \mathbb{C}$ be analytic and let $Z_t = X_t + iY_t$ where (X, Y) is a Brownian motion in $\mathbb{R}^2$. Use Itô's formula to show that $M = f(Z)$ is a local martingale in $\mathbb{R}^2$. Show further that M is a time-change of Brownian motion in $\mathbb{R}^2$.

ii. Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and fix $z \in \mathbb{D}$. What is the hitting distribution for Z on ∂D in the case that $Z_0 = 0$? By applying a Möbius transformation $\mathbb{D} \to \mathbb{D}$ and using the previous part, determine the hitting distribution for Z on $\partial\mathbb{D}$.

Problem 4 (Gronwall's Lemma)**.** Let $T > 0$ and let f be a non-negative, bounded, measurable function on $[0, T]$. Suppose that there exist $a, b \geq 0$ such that

$$f(t) \leq a + b\int_0^t f(s)ds \quad \text{for all} \quad t \in [0, T]. \tag{1}$$

Show that $f(t) \leq ae^{bt}$ for all $t \in [0, T]$.