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Problem 1. Suppose that $(Z_t)_{t\geq 0}$ is a continuous local martingale which is strictly positive almost surely. Show that there is a unique continuous local martingale M such that $Z = \mathcal{E}(M)$, where

$$\mathcal{E}(M)_t = \exp(M_t - \tfrac{1}{2}\langle M\rangle_t).$$

Problem 2. Let M be a continuous local martingale with $M_0 = 0$. For any $a, b > 0$, show that

$$P\left(\sup_{t\geq 0} M_t \geq a, \langle M\rangle_\infty \leq b\right) \leq \exp\left(-\frac{a^2}{2b}\right).$$

Problem 3. Suppose that X is a continuous local martingale with quadratic variation

$$\langle X\rangle_t = \int_0^t A_s ds$$

for a non-negative, previsible process $(A_t)_{t\geq 0}$. Show that there exists a Brownian motion B (possibly defined on a larger probability space) such that

$$X_t = \int_0^t A_s^{1/2} dB_s.$$

Hint: Let $H_t = A_t^{-1/2} 1[A_t \neq 0]$, let W be any Brownian motion defined on the same probability space, and define

$$B_t = \int_0^t H_t dX_t + \int_0^t 1[A_s = 0] dW_s.$$

Problem 4 (The Reflection Principle Revisited)**.** Using the results of this course, give a *short* proof of the reflection principle: if B is a standard Brownian motion relative to a filtration $(\mathcal{F}_t)_{t\geq 0}$, and T is a stopping time for the same filtration, then

$$W_t = \begin{cases} B_t & t \leq T; \\ 2B_T - B_t & t > T. \end{cases}$$

is also a standard Brownian Motion.