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Problem 1. Let H be a Hilbert space and $D \subset H$ a subset. Show that D is dense iff:

$$(f, d) = 0 \qquad \text{for every } d \in D$$

implies that $f = 0$.

Problem 2. Let $M, N \in \mathrm{Mart}_c^2$.

i. Show that $\langle M, N\rangle$ is the unique continuous adapted process that is locally of bounded variation such that $MN - \langle M, N\rangle$ is a martingale and $\langle M, N\rangle = 0$.

ii. Show that the map $(M, N) \mapsto \langle M, N\rangle_t$ is bilinear and symmetric.

Extend these properties to local and semimartingales.

Problem 3. Let τ be a stopping time.

i. Show that

$$\langle M^\tau, N^\tau\rangle_t = \langle M^\tau, N\rangle_t = \langle M, N\rangle_{t\wedge\tau}.$$

ii. Show that

$$\int_0^\infty \alpha 1_{[0,\tau]}\, dM = \int_0^\tau \alpha\, dM = \int_0^\infty \alpha\, dM^\tau.$$

Problem 4. Let X be a continuous semimartingale and α a locally bounded left-continuous adapted process. Show that for any nested partition $(t_i^m)_i$ with step size going to 0:

$$\int_0^t \alpha\, dX = \lim_{m\to\infty} \sum_{i=1}^{n_m} \alpha_{t_{i-1}^m} (X_{t_i^m} - X_{t_{i-1}^m}).$$