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Problem 1 (Wiener integral I)**.** Let $f \in L^2(0,\infty)$ and verify the following properties of the Wiener integral $X_t = \int_0^t f(s)dB_s$:

i. X can almost surely be defined as a continuous process;

ii. X is a martingale;

iii. $X_t^2 - \int_0^t f(s)^2\, ds$ is a martingale.

Problem 2 (Wiener integral II)**.** A real-valued process (X_t) is Gaussian if for any finite family $0 \le t_1 < t_2 < \cdots < t_n < \infty$, the random vector $(X_{t_1}, \dots, X_{t_n})$ is Gaussian. Let $f \in L^2(0,\infty)$ and show that the Wiener integral $X_t = \int_0^t f(s)dB_s$ is Gaussian, and compute its covariance.

Problem 3 (Liouville's Theorem in dimension two)**.** Suppose that $u : \mathbb{R}^2 \to \mathbb{R}$ is bounded and harmonic, and fix $x, y \in \mathbb{R}^2$. Let B be a Brownian motion; for $\epsilon > 0$, let

$$T_\epsilon = \inf\{t \ge 0 : |B_t - y| \le \epsilon\}.$$

By considering $\mathbb{E}_x\left(u(B_{T_\epsilon})\right)$, show that $u(x) = u(y)$, and conclude that u is constant.

Problem 4 (Transcience of Brownian motion in dimension three)**.** Complete of the proof of the statement in class that Brownian motion is transient in $d = 3$. What about $d \ge 3$?