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Problem 1. Let B be a standard Brownian motion. Show that the following process are martingales:

$$B_t^2 - t, \qquad e^{\lambda B_t - \frac{1}{2}\lambda^2 t}$$

where $\lambda \in \mathbb{R}$ is a constant.

Problem 2. Let B be a standard Brownian motion and fix $t \geq 0$. For $n \geq 1$, let $\Delta_n = \{0 : t_0(n) < t_1(n) < \cdots < t_{m_n}(n) = t\}$ be a partition of $[0,t]$ such that

$$h_n = \max_{1 \leq i \leq m_n} \big(t_i(n) - t_{i-1}(n)\big) \to 0 \quad \text{as} \quad n \to \infty.$$

Show that

$$\langle B \rangle_t^n = \sum_{i=1}^{m_n} (B_{t_i} - B_{t_{i-1}})^2 \to t \quad \text{in } L^2. \tag{1}$$

Show that if the subdivision is dyadic, then the convergence is also almost sure.

Problem 3. Let B be a standard Brownian motion and let

$$\widehat{B}_t = B_t - \int_0^t \frac{B_s}{s} ds.$$

i. Show that $\widehat{B}$ is not a martingale in the filtration generated by B.

ii. Show that $\widehat{B}$ is a continuous Gaussian process and identify its mean and covariance. Hence show that $\widehat{B}$ is a martingale in its own filtration.

Here a process X_t is Gaussian if for any $t_1 < \cdots < t_n$ the random vector $(B_{t_k})_k$ is jointly Gaussian.

Problem 4 (Law of the Iterated Logarithm)**.** Let $(B_t)_{t \geq 0}$ be a standard Brownian motion starting at 0, and for $t \geq 0$, let

$$S_t = \sup_{s \leq t} B_s. \tag{2}$$

i. Fix $\epsilon > 0$, and consider $t_n = (1+\epsilon)^n$. Show that, almost surely,

$$S_{t_n} \leq (1+\epsilon)\sqrt{2 t_n \log\log t_n} \qquad \text{for all } n \text{ large enough.} \tag{3}$$

Hence, show that

$$\limsup_{t \to \infty} \frac{S_t}{\sqrt{2t \log\log t}} \leq 1 \qquad \text{almost surely.} \tag{4}$$

ii. Let $\theta > 1$, $t_n = \theta^n$, and fix $0 < \alpha < \sqrt{1 - \theta^{-1}}$. Show that, almost surely,

$$B_{t_n} - B_{t_{n-1}} \geq \alpha \sqrt{2 t_n \log\log t_n} \qquad \text{infinitely often.} \tag{5}$$

Conclude that

$$\limsup_{t \to \infty} \frac{S_t}{\sqrt{2t \log\log t}} \geq 1 \qquad \text{almost surely.} \tag{6}$$

iii. Finally, deduce that

$$\limsup_{t \to 0} \frac{B_t}{\sqrt{2t \log\log \frac{1}{t}}} = 1 \qquad \text{almost surely.} \tag{7}$$