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Problem 1. Extend the Garsia-Rodemich-Rumsey inequality and its corollary to $t, s \in \mathbb{R}^d$. For this define B as in the lecture except that the integrals over t, s range over $B_T(0) \subset \mathbb{R}^d$. Then adapt the proof to show

$$|x(t) - y(s)| \leq C \int_0^{|t-s|} \psi^{-1}(cB/u^{2d}) p(u)\, du$$

where C, c are d-dependent constants.

Problem 2. Informally, Brownian motion is the Gaussian process with covariance $(-\Delta)^{-1}$ on $[0, \infty)$ where Δ has Dirichlet boundary conditions at 0. It should correspond to the illdefined Gaussian measure

$$\text{“}\exp\left(-\frac{1}{2}(B, -\Delta B)\right) \prod_t dB_t\text{”}$$

The rigorous implementation of this picture is to define Brownian motion (as the continuous modification) of the Gaussian process indexed by H^* defined as follows. Let H denote the Hilbert space of absolutely continuous $h : [0, \infty) \to \mathbb{R}$, i.e.,

$$h(t) = \int_0^t h'(s)\, ds,$$

such that the following H^1 norm is finite:

$$\|h\|_H = \left(\int_0^\infty h'(t)^2\, dt\right)^{1/2}.$$

This space H is called the Cameron–Martin space of Brownian motion. Show that its dual space H^* can be identified with the completion of

$$\{\mu \in M[0, \infty) : \int_0^\infty (s \wedge t)\, \mu(ds)\, \mu(dt) = (\mu, \mu)_{H^*} < \infty, \mu(\{0\}) = 0\},$$

where M is the space of signed measures. The identification of μ with a bounded linear functional $l_\mu : H \to \mathbb{R}$ is

$$l_\mu(h) = (\mu, h) = \int_0^\infty h(s)\mu(ds)$$

with norm $\|\mu\|_{H^*} = \sup_{\|h\|_H \leq 1}(\mu, h)$. Brownian motion can then be defined in terms of a Gaussian Hilbert space indexed by H^*. This implements the notion that the covariance is $(-\Delta)^{-1}$ since for nice h and μ:

$$(h, h)_H = (h, -\Delta h), \qquad (\mu, \mu)_{H^*} = (\mu, (-\Delta)^{-1}\mu).$$

Problem 3 (Brownian bridge)**.** Let $-\Delta$ be the Laplacian on $[0, 1]$ with Dirichlet boundary conditions at 0 and 1. Show that its inverse is given by

$$Kh(t) = \int_0^1 (1_{s \leq t} s(1-t) + 1_{s \geq t} t(1-s)) h(s)\, ds.$$

and define the associated Gaussian Hilbert space. Show the process has a continuous modification.

Problem 4. The Gaussian free field (GFF) with mass 1 on the torus $\mathbb{T}^d = [0,1]^d/\sim$ is a Gaussian process with covariance $(-\Delta+1)^{-1}$. Realize it as in terms of a Gaussian Hilbert space indexed by $H^{-1}(\mathbb{T}^d)$ so formally

$$\mathbb{E}\big[(\Phi, f)(\Phi, g)\big] = (f, (-\Delta+1)^{-1}g).$$

By considering the Fourier basis, show that $H^{-1}(\mathbb{T}^d)$ has Hilbert–Schmidt embedding into $H^{-s}(\mathbb{T}^d)$ when $s > (d-2)/2$. This means that if (f_k) is an orthnonormal basis of $H^{-1}(\mathbb{T}^d)$ then

$$\sum_k \|f_k\|^2_{H^{-s}(\mathbb{T}^d)} < \infty.$$

Here

$$\|f\|^2_{H^{-s}} = \|(-\Delta+1)^{-s/2} f\|^2_{L^2} = \int \frac{|\hat{f}(p)|^2}{(|p|^2+1)^{s/2}} \frac{dp}{(2\pi)^d}.$$

Deduce that the Φ can be realized as a process assuming values in $H^{-s}(\mathbb{T}^d)$ almost surely.

Problem 5. The GFF on $\mathbb{R}^d$ is defined analogously as the Gaussian process with covariance $(-\Delta+1)^{-1}$. Realize it it as in terms of a Gaussian Hilbert space indexed by $H^{-1}(\mathbb{R}^d)$. You may now use without proof that $H^{-1}(\mathbb{R}^d)$ has Hilbert–Schmidt embedding into the weighted Sobolev space $H^{-s,r}(\mathbb{R}^d)$ with norm

$$\|f\|_{H^{-s,r}} = \|(1+x^2)^{-r/2}(-\Delta+1)^{-s/2} f\|_{L^2}$$

for suitable $s, r > 0$. Deduce that Φ can be realized as a process assuming values in $H^{-s,r}(\mathbb{R}^d)$ almost surely. In particular, Φ is a random Schwartz distribution almost surely.

Problem 6. The Besov–Hölder space $C^{-\gamma}(\mathbb{R}^d)$ can be defined as distributions with $\|f\|_{C^{-\gamma}} < \infty$ where

$$\|f\|_{C^{-\gamma}} = \sup_{0\le r\le 1} r^\gamma \|\psi_r * f\|_\infty$$

and ψ is any fixed $\psi \in C_c^\infty(\mathbb{R}^d)$ with support in $B_1(0)$ and $\psi_r(x) = r^{-d}\psi(x/r)$. The following inequality is a negative regularity analogue of the Garsia-Rodemich-Rumsey inequality:

$$\|f\|_{C^{-\gamma}} \le C \left(\int_0^1 r^{p\gamma - d} \int |\psi_r * f(x)|^p \, dx \, \frac{dr}{r} \right)^{1/p}.$$

Assuming this inequality, show that for $d \ge 2$ the Gaussian free field Φ satisfies $\chi\Phi \in C^{-\gamma}$ for $\gamma > (d-2)/2$ for any $\chi \in C_c^\infty(\mathbb{R}^d)$. You may restrict to $d = 3$ or $d = 2$ to simplify computations.