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Problem 1. Let $f\colon [0,\infty) \to \mathbb{R}$ be absolutely continuous, in the sense that

$$f(t) = f(0) + \int_0^t f'(s)ds \quad \text{for all} \quad t \geq 0$$

for an integrable function f'. Let $v_f(0,t)$ be the total variation of f on $(0,t]$. Show that

$$v_f(0,t) = \int_0^t |f'(s)|\, ds.$$

Problem 2. Let $f, g\colon [0,\infty) \to \mathbb{R}$ be bounded and measurable, and let $a\colon [0,\infty) \to \mathbb{R}$ be continuous and of finite variation. Show that

$$f \cdot (g \cdot a) = (fg) \cdot a$$

where $\cdot$ denotes the Lebesgue-Stieltjes integral.

Problem 3. Let $(\mathcal{F}_t)_{t\geq 0}$ be a filtration, let T be a stopping time, and let

$$\mathcal{F}_T = \{A \in \mathcal{F} : A \cap \{T \leq t\} \in \mathcal{F}_t \quad \forall t \geq 0\}\,.$$

i. Show that $\mathcal{F}_T$ is a σ-algebra.

ii. Show that T is $\mathcal{F}_T$-measurable.

iii. Suppose that X is a càdlàg, adapted process. Show that X_T is $\mathcal{F}_T$-measurable.

Problem 4. Let $(T_n)_{n\geq 1}$ denote a sequence of stopping time for a filtration $(\mathcal{F}_t)_{t\geq 0}$.

i. Show that $T^\star = \sup_n T_n$ is a stopping time for $(\mathcal{F}_t)_{t\geq 0}$.

ii. Show a random variable T is a stopping time for the filtration $\mathcal{F}_{t^+} = \bigcap_{s>t} \mathcal{F}_s$ if, and only if,

$$\{T < t\} \in \mathcal{F}_{t^+}$$

for all $t \geq 0$.

iii. Show that $T_\star = \inf_n T_n$ is a stopping time for $(\mathcal{F}_{t^+})_{t\geq 0}$.

Problem 5 (Stochastic Calculus of a Total Variation Processes)**.** Let T and ξ denote two independent random variables defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with

$$\mathbb{P}(T \leq t) = t \;\text{ for } t \in [0,1], \qquad \mathbb{P}(\xi = 1) = \mathbb{P}(\xi = -1) = 1/2.$$

Define $X_t = \xi \mathbf{1}_{t\geq T}$ and $\mathcal{F}_t = \sigma(X_s : s \leq t)$. Show that X is a martingale with respect to $(\mathcal{F}_t)_{t\in[0,1]}$, and that it is of finite variation. For bounded processes H, define pathwise

$$Y_t(\omega) := \int_{(0,t]} H_s(\omega) dX_s(\omega) \qquad \text{for all } \omega \in \Omega,$$

where the right-hand side is a Lebesgue-Stieltjes integral. Verify that, if H is a simple process

$$H_t = a_u \mathbf{1}_{t\in(u,v]}, \qquad a_u \in L^\infty(\mathcal{F}_u), \quad 0 \leq u < v \leq 1,$$

then (Y_t) is a martingale; use a monotone class argument to extend this to bounded H in the predictable sigma-field, which is generated by all left-continuous processes or simple H as above. What happens if we take $H = X$?

Problem 6. Suppose that $X \in L^1(\Omega, \mathcal{F}, \mathbb{P})$. Show that the family

$$\mathcal{X} = \{\mathbb{E}[X \mid \mathcal{G}] : \mathcal{G} \subseteq \mathcal{F} \quad \text{is a } \sigma\text{-algebra}\} \qquad \text{is UI.}$$

Problem 7 (A silly martingale)**.** Construct a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, a L^∞-bounded martingale $(M_t)_{t=0}^1$ without the condition of being right-continuous and a stopping time T taking values in $[0, 1]$, such that

$$\mathbb{E}(M_T) \neq \mathbb{E}(M_0).$$