Roland Bauerschmidt ⟨rb812@cam.ac.uk⟩, *Daniel Heydecker* ⟨dh489@cam.ac.uk⟩ *Lent 2020*

Problems marked with (†) may be handed in for marking (CCA pidgeonhole G/H). Problems marked with (⋆) are additional questions

Problem 1. Let $\mathbb{P}$ and $\tilde{\mathbb{P}}$ be probability measures on the same space such that $\mathbb{P} \ll \tilde{\mathbb{P}}$.

i. Show that if Z_n, Z are random variables such that $Z_n \to Z$ in $\mathbb{P}$-probability, then $Z_n \to Z$ in $\tilde{\mathbb{P}}$-probability.

ii. Let X be a continuous semimartingale under both $\mathbb{P}$ and $\tilde{\mathbb{P}}$. Show that X has the same quadratic variation process under both measures.

Problem 2. (†) Let b be bounded and measurable. Use Girsanov's theorem to construct a weak solution to the SDE

$$dX_t = b(X_t)dt + dB_t$$

over the finite (non-random) time interval $[0, T]$.

Problem 3. (†) Show that the SDE

$$dX_t = 3\mathrm{sign}(X_t)|X_t|^{1/3}dt + 3|X_t|^{2/3}dB_t, \quad X_0 = 0$$

has strong existence but not pathwise uniqueness.

Problem 4. Find the unique strong solution to the SDE

$$dX_t = \frac{1}{2}X_t dt + \sqrt{1 + X_t^2}dB_t, \quad X_0 = x.$$

(Hint: consider the change of variables $Y_t = \sinh^{-1}(X_t)$.)

Problem 5. (†) Construct a filtered probability space on which a Brownian motion B and an adapted process X are defined and such that

$$X_t = \int_0^t \frac{X_s}{s} ds + B_t, \quad B_0 = X_0 = 0.$$

Is X adapted to the filtration generated by B? Is B a Brownian motion in the filtration generated by X?

Problem 6. Let X be a solution of the SDE

$$dX_t = X_t g(X_t) dB_t$$

where g is bounded and $X_0 = x > 0$ is non-random.

i. By applying Ito's formula to

$$X_t \exp\left(-\int_0^t g(X_s)dB_s + \frac{1}{2}\int_0^t g^2(X_s)ds\right)$$

show that $\mathbb{P}(X_t > 0 \text{ for all } t \geq 0) = 1$.

ii. Show that $\mathbb{E}(X_t) = X_0$ for all $t \geq 0$.

iii. Fix a non-random time horizon $T > 0$. Show that there exists a measure $\widehat{\mathbb{P}}$ on $(\Omega, \mathcal{F}_T)$ which is mutually absolutely continuous with respect to $\mathbb{P}$ and a $\widehat{\mathbb{P}}$-Brownian motion $\widehat{B}$ such that

$$dY_t = Y_t g(1/Y_t) d\widehat{B}_t$$

where $Y_t = 1/X_t$.

Problem 7. Consider the Cauchy problem for the quasi-linear parabolic equation

$$\frac{\partial V}{\partial t} = \frac{1}{2}\Delta V - \frac{1}{2}|\nabla V|^2 + k \quad \text{on} \quad (0,\infty)\times\mathbb{R}^d,$$

with $V(0,x) = 0$ for $x \in \mathbb{R}^d$ where $k\colon \mathbb{R}^d \to [0,\infty)$ is a continuous function. Suppose also that $V\colon [0,\infty)\times\mathbb{R}^d \to \mathbb{R}$ is continuous on its domain, of class $C^{1,2}$ on $(0,\infty)\times\mathbb{R}^d$, and satisfies the quadratic growth condition for every $T > 0$:

$$-V(t,x) \le C + a|x|^2, \quad (t,x)\in[0,T]\times\mathbb{R}^d, \quad a < \frac{1}{2T}.$$

Show that $V(t,x)$ is given by

$$V(t,x) = -\log \mathbb{E}_x\left[\exp\left(-\int_0^t k(W_s)ds\right)\right]$$

for $t \ge 0$ and $x \in \mathbb{R}^d$.

Problem 8. Let $b\colon \mathbb{R}^d \to \mathbb{R}$ and $\sigma\colon \mathbb{R}^d \to \mathbb{R}^{d\times d}$ be bounded and continuous. For each n, j, set $t_j^n = n2^{-j}$ and $\psi_n(t) = t_j^n$ if $t \in [t_j^n, t_{j+1}^n)$. Assume that (X_0^n) is a tight sequence, and that X^n solves

$$X_t^n = X_0^n + \int_0^t b\left(X^n_{\psi_n(u)}\right)du + \int_0^t \sigma\left(X^n_{\psi_n(u)}\right)dB_u. \tag{1}$$

Show that for each $m, T > 0$ there exists a constant $C > 0$ such that

$$\mathbb{E}[|X_t^n - X_s^n|^{2m}] \le C(t-s)^m \quad \text{for all} \quad 0 \le s < t \le T. \tag{2}$$

Explain what it means for the sequence (X^n) to be tight in the space $C([0,T],\mathbb{R}^d)$. By looking at the proof of Kolmogorov's continuity criterion, explain why (2) implies that (X^n) is tight.

Problem 9. Consider the SDE

$$dX_t = X_t^2 dB_t. \tag{$\star$}$$

i. Show that, if $(X_t)_{t\ge0}$ and $(Y_t)_{t\ge0}$ are two (globally-defined) solutions to ($\star$) with the same starting point x_0, then they have the same law.

ii. By considering the process $\widetilde{X}_t = 1/|B_t - \xi|$ where B is a three-dimensional Brownian motion and ξ is a standard Gaussian in $\mathbb{R}^3$ independent of B, show that the SDE has a weak solution.

iii. Let $\Phi(s) = \int_{-\infty}^s e^{-t^2/2}\,dt/\sqrt{2\pi}$ be the Gaussian distribution function. Verify that both

$$u^1(t,x) = x\left(2\Phi(1/(x\sqrt{t})) - 1\right) \quad \text{and} \quad u^2(x,t) = x$$

solve the PDE

$$\frac{\partial u}{\partial t} = \frac{x^4}{2}\cdot\frac{\partial^2 u}{\partial x^2}, \quad u(0,x) = x \qquad \text{on } (0,\infty)\times(0,\infty).$$

iv. Which of these solutions corresponds to $u(t,x) = \mathbb{E}_x(X_t)$?

Problem 10*. Consider the SDE

$$dX_t = -X_t^3 dt + dB_t; \qquad X_0 = x_0. \tag{$\star$}$$

Recall that there exists a unique maximal solution $(X_t)_{t<\zeta}$ to ($\star$).

i. Define $T = \inf\{t \ge 0 : X_t = 0\}$. Show that $X_t \le x_0 + B_t$ for all $t \le T \wedge \zeta$ and deduce that

$$\mathbb{P}_{x_0}(T_0 < \zeta) = 1. \tag{3}$$

ii. Hence show that there exists a sequence of a.s. finite stopping times $T_0 < S_1 < T_1 < \ldots < S_n < T_{n+1} < S_{n+1} < .. < \zeta$ such that $X_{T_n} = 0$ and $|X_{S_n}| = 1$ for all n.

iii. Conclude that $\zeta = \infty$ almost surely, so that the solution to ($\star$) is defined for all $t \ge 0$.