Stochastic Calculus and Applications (Lent 2019)

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Primary references:
J.-F. Le Gall, Brownian Motion, Martingales, and Stochastic Calculus, Springer
D. Revuz, M. Yor, Continuous Martingales and Brownian Motion, Springer
Past Cambridge lecture notes (N. Berestycki, J. Miller, V. Silvestri, M. Tehranchi, ...)

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1. Introduction

1.1. Motivation

ODE: $\dot{x}(t) = F(x(t))$ — fundamental in analysis

SDE: $\dot{x}(t) = F(x(t)) + \eta(t)$

random noise

What should η be?

- For $|t-s| \gg 0$, $\eta(t)$ and $\eta(s)$ should be essentially independent.
- Idealisation: $\eta(t)$ and $\eta(s)$ should be independent for $t \neq s$.

Such an η exists, White Noise, but it is only a random generalised function (random Schwartz distribution).

But even if $F=0$, to make sense of

$$\dot{x} = \eta, \quad \text{i.e.,} \quad x(t) - x(0) = \int_0^t \eta(s)\, ds, \qquad (*)$$

deterministically, η should at least be a signed measure.

- White Noise is not a random signed measure.
- If $(*)$ held, for any $0 = t_0 < t_1 < t_2 \cdots$, the increments

$$x(t_i) - x(t_{i-1}) = \int_{t_{i-1}}^{t_i} \eta(s)\, ds$$

should be independent and their variance should be proportional to $|t_i - t_{i-1}|$ (by subdividing).

$\Rightarrow$ x should be Brownian Motion.

In which sense can we make sense of this?

1.2. The Wiener Integral

<u>Defn</u>. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Then $S \subset L^2(\Omega, \mathcal{F}, \mathbb{P})$ is a <u>Gaussian space</u> if S is a closed linear subspace and any $X \in S$ is a centred Gaussian random variable.

<u>Example</u>. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space on which a sequence of independent random variables $X_i \sim \mathcal{N}(0,1)$ is defined. Then the X_i are an orthonormal system in $L^2(\Omega, \mathcal{F}, \mathbb{P})$:

$$\mathbb{E}(X_i X_j) = 0 \quad \text{for all } i \neq j, \quad \mathbb{E} X_i^2 = 1,$$

and $S = \overline{\operatorname{span}\{X_i\}}$ is a Gaussian space.
(Exercise: the limit in L^2 of Gaussian random variables is Gaussian.)

<u>Prop</u>. Let H be a separable Hilbert space and $(\Omega, \mathcal{F}, \mathbb{P})$ as in the example. Then there is an isometry $I : H \to S$. In particular, for every $f \in H$, there is a random variable $I(f) \in S$ s.t.

$$I(f) \sim \mathcal{N}(0, (f,f)_H)$$

and $$\mathbb{E}(I(f)\, I(g)) = (f,g)_H.$$

Moreover, $I(\alpha f + \beta g) = \alpha I(f) + \beta I(g)$ a.s.

<u>Proof</u>. Let $(e_i)_{i=1}^{\infty}$ be an orthonormal Hilbert basis for H. For $f \in H$, set

$$I(f) = \sum_{i=1}^{\infty} (f, e_i) X_i \in L^2(\Omega, \mathcal{F}, \mathbb{P}).$$

The limit exists in L^2 since $\sum_{i=1}^{k}(f,e_i)X_i$ is a Cauchy sequence:

$$\mathbb{E}\left(\sum_{i=1}^{k}(f,e_i)X_i - \sum_{i=1}^{\ell}(f,e_i)X_i\right)^2 \le \sum_{i=k}^{\ell}|(f,e_i)|^2 \to 0 \quad \text{since } f \in H.$$

(In fact, $k \mapsto \sum_{i=1}^{k}(f,e_i)X_i$ is also a martingale, so the limit also exists almost surely.)

The map I is an isometry since it maps the orthonormal basis (e_i) to the orthonormal system (X_i) in $L^2(\Omega, \mathcal{F}, \mathbb{P})$.

<u>Defn</u>. A Gaussian <u>White Noise</u> on $\mathbb{R}_+$ is an isometry WN from $L^2(\mathbb{R}_+)$ into a Gaussian space. For $A \subset \mathbb{R}_+$ Borel, write $WN(A) = WN(\mathbb{1}_A)$.

<u>Prop</u>. (i) For $A \subset \mathbb{R}_+$ Borel with $|A| < \infty$, $WN(A) \sim \mathcal{N}(0, |A|)$.

(ii) If $A, B \subset \mathbb{R}_+$ Borel, $A \cap B = \emptyset$ then $WN(A)$ and $WN(B)$ are independent.

(iii) If $A = \bigcup_{i=1}^{\infty} A_i$ for disjoint sets A_i with $|A_i| < \infty$ and $|A| < \infty$, then

$$WN(A) = \sum_{i=1}^{\infty} WN(A_i) \quad \text{in } L^2 \text{ and a.s.} \qquad (*)$$

<u>Proof</u>. (i) holds since $(\mathbb{1}_A, \mathbb{1}_A) = |A|$, (ii) holds since $E(WN(A)\,WN(B)) = 0$ and uncorrelated jointly Gaussian random var. are independent.

(iii) $M_n = \sum_{i=1}^{n} WN(A_i)$ is a martingale bounded in L^2:

$$\mathbb{E} M_n^2 \underset{\uparrow}{=} \sum_{i=1}^{n} \mathbb{E}\, WN(A_i)^2 = \sum_{i=1}^{n} |A_i| < |A|$$

disjointness: $\mathbb{E} WN(A_i)\, WN(A_j) = 0$ for $i \neq j$

Thus $\sum_{i=1}^{\infty} WN(A_i)$ converges a.s. and in L^2. Similarly,
$E\left(\left(WN(A) - \sum_{i=1}^{n} WN(A_i)\right)^2\right) \to 0$, so (*) holds.

Rk. WN looks like a random measure, $A \in \mathcal{B}(\mathbb{R}_+) \mapsto WN(\omega, A)$, where $\omega \in \Omega$, but it is not!
In (*), the event $E \subset \Omega$ of ω for which (*) holds depends on the sets A_i.

For $t \geq 0$, define $B_t = WN([0,t])$.
Fact. For any $t_1, \dots, t_n$, the vector $(B_{t_i})_i$ is jointly Gaussian and

$$E(B_s B_t) = s \wedge t \quad \text{for all } s,t \geq 0.$$

Moreover, $B_0 = 0$ a.s. and $B_t - B_s$ is independent of $\sigma(B_r, r \leq s)$
$$B_t - B_s \sim \mathcal{N}(0, t-s) \quad \text{for } t \geq s.$$

Example. Let $f \in L^2(\mathbb{R}_+)$ be a step function: $f = \sum_{i=1}^{n} f_i 1_{[t_i, t_{i+1}]}$, $t_i < t_{i+1}$.
Then

$$WN(f) = \sum_{i=1}^{n} f_i (B_{t_{i+1}} - B_{t_i})$$

This motivates the notation

$$WN(f) = \int f(s)\, dB_s.$$

If (B) was a function of bounded variation, for a.e. $\omega \in \Omega$, the last line could be defined as a Lebesgue-Stieltjes integral.

Prop (→A.P). There is a modification of (B_t) s.t. $t \mapsto B_t$ is continuous, almost surely.

Defn. This modification is called Brownian motion.

1.3. The Lebesgue-Stieltjes integral

For an interval $I \subset \mathbb{R}$, we always use the Borel σ-algebra unless otherwise stated.

Defn. Let $T>0$.

- A signed measure μ on $[0,T]$ is the difference of two finite positive measures $\mu_\pm$ on $[0,T]$ with disjoint support. The decomposition $\mu = \mu_+ - \mu_-$ is called the Hahn-Jordan decomposition of μ.
- The total variation of a signed measure $\mu = \mu_+ - \mu_-$ is the positive measure $|\mu| = \mu_+ - \mu_-$.

Prop (Hahn-Jordan). For any positive measures μ_1 and μ_2 on $[0,T]$, there is a signed measure μ s.t. $\mu = \mu_1 - \mu_2$.

Proof. Let $\nu = \mu_1 + \mu_2$. By the Radon-Nikodym Theorem, there are Borel functions $f_i \geq 0$ on $[0,T]$ s.t.

$$\mu_i(dt) = f_i(t)\,\nu(dt).$$

Let $f(t) = f_1(t) - f_2(t)$. Then

$$(\mu_1 - \mu_2)(dt) = f(t)\,\nu(dt) = \overbrace{f(t)^+\,\nu(dt)}^{\mu_+(dt)} - \overbrace{f(t)^-\,\nu(dt)}^{\mu_-(dt)}.$$

where $f(t)^+ = f(t) \vee 0$, $f(t)^- = -f(t) \wedge 0$ are the positive and negative parts of $f(t)$. This gives the decomposition into disjoint measures.

Defn. Let $T \geq 0$.

- A function $a : [0,T] \to \mathbb{R}$ is càdlàg, for which we also write $a \in D([0,T])$, if $a(t_+) = a(t)$ for all t and $a(t_-)$ exists for all t.
  Here $a(t_\pm) = \lim_{s \to 0\pm} a(t+s)$.

- The total variation of a function $a : [0,T] \to \mathbb{R}$ is

$$V_a(0,T) = \sup\left\{ \sum_{i=1}^{n} |a(t_i) - a(t_{i-1})| : 0 \leq t_0 \leq t_1 < \cdots \leq t_n \leq T \right\}.$$

- A function $a : [0,T] \to \mathbb{R}$ is of bounded variation, for which we write $a \in BV([0,T])$, if $V_a(0,T) < \infty$.

Prop.

(i) Let μ be a signed measure on $[0,T]$. Then $a(t) = \mu([0,t])$ is càdlàg and $|\mu|((0,t]) = V_a(0,t)$, i.e., $|\mu|([0,t]) = |a(0)| + V_a(0,t)$. In particular, $a \in BV([0,T])$.

(ii) Let $a : [0,T] \to \mathbb{R}$ be càdlàg and of bounded variation. Then there is a signed measure μ s.t. $a(t) = \mu([0,t])$.

To prove the proposition, we assume the following fact from measure theory.

Fact. The map $\nu \mapsto f$, $f(t) = \nu[0,t]$ (distribution function) is a bijection from finite positive measures on $[0,T]$ to increasing right-continuous functions. (Such f are càdlàg.)

Proof of proposition. (i) Let $\mu = \mu_+ - \mu_-$ be the Hahn-Jordan decomposition of μ. Then

$$a(t) = \underbrace{\mu_+([0,t])}_{a_+(t)} - \underbrace{\mu_-([0,t])}_{a_-(t)} \quad \text{is càdlàg}$$

since $a_\pm$ are increasing right-continuous functions.

Claim: $V_a(0,t) \leq |\mu|((0,t])$.

For any subdivision $0 = t_0 < t_1 < \cdots < t_n = t$,

$$\sum_{i=1}^{n} |a(t_i) - a(t_{i-1})| = \sum_{i=1}^{n} |\mu((t_{i-1}, t_i])| \leq |\mu|((0,t]).$$

$$\Rightarrow \quad V_a(0,t) \leq |\mu|((0,t]).$$

Claim: For any nested sequence of subdivisions

$$0 = t_0^{(m)} < t_1^{(m)} < \cdots < t_{n_m}^{(m)} = t \quad \text{with } \max_i |t_i^{(m)} - t_{i-1}^{(m)}| \to 0,$$

one has

$$|\mu|((0,t]) = \lim_{m\to\infty} \sum_{i=1}^{n_m} |a(t_i^{(m)}) - a(t_{i-1}^{(m)})|.$$

In particular, $V_a(0,t) \geq |\mu|(0,t]$.

Consider the probability measure $P(ds) = \dfrac{|\mu|(ds)}{|\mu|((0,t])}$ on $(0,t]$.

Let $\mathcal{F}_m = \sigma((t_{i-1}^{(m)}, t_i^{(m)}],\ 1 \leq i \leq n_m)$. Note that $\mathcal{F}_{m+1} \supset \mathcal{F}_m$.

Let $X = \dfrac{d\mu}{d|\mu|} = 1_{\operatorname{supp}\mu_+} - 1_{\operatorname{supp}\mu_-}$ and let $X_m = \mathbb{E}(X \mid \mathcal{F}_m)$.

For $s \in (t_{i-1}^{(m)}, t_i^{(m)}]$, then

$$X_m(s) = \frac{\mu((t_{i-1}^{(m)}, t_i^{(m)}])}{|\mu|((t_{i-1}^{(m)}, t_i^{(m)}])} = \frac{a(t_i^{(m)}) - a(t_{i-1}^{(m)})}{|\mu|((t_{i-1}^{(m)}, t_i^{(m)}])}$$

Since (X_m) is a bounded martingale, also $X_m \to Y$ in L^1 and a.s., for some Y.

Since $\bigvee \mathcal{F}_m = \mathcal{B}((0,t])$, it follows that $X = Y$ a.s.

$\Rightarrow \mathbb{E}|X_m| \to \mathbb{E}|X| = 1$

$$\Leftrightarrow \frac{1}{|\mu|((0,t])} \sum_{i=1}^{n_m} |a(t_i^{(m)}) - a(t_{i-1}^{(m)})| \longrightarrow 1$$

which is the claim.

(ii) Let a be as in (ii). It also suffices to assume $a(0)=0$. Let

$$a_\pm(t) = \frac{1}{2}\left(V_a(0,t) \pm a(t)\right).$$

Claim: $a_\pm$ are increasing.

Let $0 = t_0 < t_1 < \cdots < t_n = t$ be a subdivision of $[0,t]$, and let $s > t$. Then $t_0 < \cdots < t_n = t < s$ is a subdivision of $[0,s]$.

$$\Rightarrow 2a_\pm(s) = V_a(0,s) \pm a(s) \geq \underbrace{\sum_{i=1}^{n} |a(t_i) - a(t_{i-1})|}_{\geq V_a(0,t) - \varepsilon} + \underbrace{|a(s) - a(t)| \pm a(s)}_{\geq \pm a(t)}$$

for suff. fine subdivision: $\geq V_a(0,t) - \varepsilon$ $\quad \geq \pm a(t)$

$\Rightarrow a_\pm(s) \geq a_\pm(t) - \varepsilon/2$ for all $\varepsilon > 0$

$\Rightarrow a_\pm(s) \geq a_\pm(t)$, i.e. $a_\pm$ are increasing

Claim (Example Sheet): V_a is right-continuous.

$\Rightarrow a_\pm$ is right-continuous

$\Rightarrow a_\pm(t) = \tilde{\mu}_\pm([0,t])$ for finite positive measures $\tilde{\mu}_\pm$

Let $\mu = \tilde{\mu}_+ - \tilde{\mu}_-$. Then μ is a signed measure and

$a(t) = a_+(t) - a_-(t) = \mu[0,t]$.

<u>Example</u>. Let $a : [0,1] \to \mathbb{R}$ be given by

$$a(t) = \begin{cases} 1 & (t < \frac{1}{2}) \\ 0 & (t \geq \frac{1}{2}). \end{cases}$$

Then $V_a(0,1) = 1$. The associated signed measure is

$\mu = \delta_0 - \delta_{\frac{1}{2}}$ with $|\mu| = \delta_0 + \delta_{\frac{1}{2}}$, $|\mu|((0,1]) = \delta_{\frac{1}{2}}((0,1]) = 1$.

Defn. Let $a : [0,T] \to \mathbb{R}$ be càdlàg of bounded variation, and let μ be the associated signed measure.
For $h \in L^1([0,T], |\mu|)$, the <u>Lebesgue-Stieltjes Integral</u> is defined by

$$\int_s^t h(s)\, da(s) = \int_{(s,t]} h(s)\, \mu(ds), \qquad 0 \le s < t \le T$$

$$\int_s^t h(s)\, |da(s)| = \int_{(s,t]} h(s)\, |\mu|(ds)$$

We also write

$$(h \circ a)(t) = \int_0^t h(s)\, da(s), \qquad 0 < t \le T$$

Fact. Let $a : [0,T] \to \mathbb{R}$ be càdlàg and BV, $h \in L^1([0,T], |da|)$.
Then

$$\left| \int_0^t h(s)\, da(s) \right| \le \int_0^t |h(s)|\, |da(s)|$$

and the function $h \circ a : [0,T] \to \mathbb{R}$ is càdlàg and BV with signed measure $h(s)\, da(s)$, $|h(s)\, da(s)| = |h(s)|\, |da(s)|$.

Prop. Let $a : [0,T] \to \mathbb{R}$ be càdlàg and BV. Let $h : [0,T] \to \mathbb{R}$ be left-continuous and bounded. Then

$$\int_0^t h(s)\, da(s) = \lim_{m\to\infty} \sum_{i=1}^{n_m} h(t_{i-1}^{(m)}) \left(a(t_i^{(m)}) - a(t_{i-1}^{(m)})\right), \quad t \le T$$

$$\int_0^t h(s)\, |da(s)| = \lim_{m\to\infty} \sum_{i=1}^{n_m} h(t_{i-1}^{(m)}) \left|a(t_i^{(m)}) - a(t_{i-1}^{(m)})\right|, \quad t \le T$$

for any sequence of subdivisions $0 = t_0^{(m)} < \cdots < t_{n_m}^{(m)} = t$ with $\max_i |t_i^{(m)} - t_{i-1}^{(m)}| \to 0$ as $m \to \infty$.

Proof. Set

$$\begin{cases} h_m(0) = 0 \\ h_m(s) = h(t_{i-1}^{(m)}) \quad \text{if } s \in (t_{i-1}^{(m)}, t_i^{(m)}] \end{cases}$$

$\Rightarrow h(s) = \lim_{m\to\infty} h_m(s)$ by left-continuity

$$\Rightarrow \lim_{m\to\infty} \sum_{i=1}^{n_m} h(t_{i-1}^{(m)})(a(t_i^{(m)}) - a(t_{i-1}^{(m)})) = \int_{[0,t]} h_m(s)\, \mu(ds) \to \int h(s)\, \mu(ds)$$

by the DCT.

The claim about $|da(s)|$ is left as an exercise.

(For nested subdivisions, proceed as in the proof of the previous proposition.)

<u>Defn</u>. A function $a:[0,\infty)\to\mathbb{R}$ is of finite variation (FV) if $a|_{[0,T]}\in BV([0,T])$ for all $T>0$.

2. Semimartingales

From now on, $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ is a filtered probability space.

Defn. A càdlàg adapted process X is a map $X: \Omega \times [0,\infty) \to \mathbb{R}$ s.t.

(i) X is càdlàg, i.e., $X(\omega, \cdot): [0,\infty) \to \mathbb{R}$ is càdlàg for all $\omega \in \Omega$

(ii) X is adapted, i.e., $X_t = X(\cdot, t)$ is $\mathcal{F}_t$-measurable for all t.

Notation: write $X \in \mathcal{F}$ to denote that a random variable X is measurable w.r.t. a σ-algebra $\mathcal{F}$.

2.1. Finite variation processes

Defn. (i) A càdlàg adapted process A is a finite variation process if $A(\omega, \cdot): [0,\infty) \to \mathbb{R}$ has finite variation for all $\omega \in \Omega$.

(ii) The total variation process V associated to a finite variation process A is

$$V_t = \int_0^t |dA_s|.$$

Fact. The total variation process V of a càdlàg adapted process is also càdlàg adapted and it is increasing.

Proof. That V is càdlàg and increasing follows from deterministic properties of the Lebesgue-Stieltjes Integral (Section 1.3).
To show that V is adapted, let $0 = t_0^{(m)} < t_1^{(m)} < \cdots < t_{n_m}^{(m)} = t$ be a nested sequence of subdivisions of $[0,t]$ with $\lim_{m\to\infty} |t_i^{(m)} - t_{i-1}^{(m)}| = 0$. Then we have seen that

$$V_t = \lim_{m\to\infty} \sum_{i=1}^{n_m} |A_{t_i^{(m)}} - A_{t_{i-1}^{(m)}}| \in \mathcal{F}_t$$

So V is indeed adapted.

Defn. Let A be a finite variation process and let H be a process s.t.

$$\forall \omega \in \Omega \ \forall t \geq 0: \int_0^t |H_s(\omega)| \, |dA_s(\omega)| < \infty.$$

Then define a process $((H \cdot A)_t)_{t\geq 0}$ by

$$(H \cdot A)_t = \int_0^t H_s \, dA_s.$$

For the process $H \cdot A$ to be adapted, we need a condition on H.

Defn. The predictable (or previsible) σ-algebra $\mathcal{P}$ is the σ-algebra on $\Omega \times [0,\infty)$ generated by the sets

$$E \times (s,t], \qquad E \in \mathcal{F}_s, \ s < t.$$

A process $H: \Omega \times [0,\infty) \to \mathbb{R}$ is previsible if it is $\mathcal{P}$-measurable.

Defn. A process $H: \Omega \times [0,\infty) \to \mathbb{R}$ is <u>simple</u>, $H \in \mathcal{E}$, if

$$H(\omega, t) = \sum_{i=1}^{n} H_{i-1}(\omega)\, 1_{(t_{i-1}, t_i]}(t)$$

for bounded random variables $H_{i-1} \in \mathcal{F}_{t_{i-1}}$ and $0 = t_0 < \cdots < t_n$.

Fact. Simple processes and their pointwise limits are predictable.

Fact. Adapted left-continuous processes are predictable.

Proof. Let H be adapted left-continuous. Then $H_t^n \to H_t$ where

$$H_t^n = \sum_{i=1}^{n2^n} H_{(i-1)2^{-n}}\, 1_{((i-1)2^{-n},\, i2^{-n}]}(t) \wedge n.$$

Since H is adapted, H^n is simple. Thus H^n is predictable. H is then also predictable as a pointwise limit of H^n.

Fact. Let H be predictable. Then $H_t \in \mathcal{F}_{t-}$ where $\mathcal{F}_{t-} = \sigma(\mathcal{F}_s : s < t)$.

Fact. Let X be adapted càdlàg. Then X_{t-} is adapted left-continuous, so predictable.

Example.

- Brownian motion is predictable since continuous.
- A Poisson process (N_t) is <u>not</u> predictable since $N_t \notin \mathcal{F}_{t-}$.

Prop. Let A be a finite variation process, and let H be a predictable process s.t. $\int_0^t |H_s||dA_s| < \infty$ for all t and ω. Then $H \cdot A$ is also a finite variation process.

Proof. By Section 1.3, for every $\omega \in \Omega$, $(H \cdot A)(\omega, \cdot)$ is of finite variation and càdlàg.

Claim: $H \cdot A$ is adapted

Consider first $H(\omega, t) = 1_{(u,v]}(t) 1_E(\omega)$, $u<v$, $E \in \mathcal{F}_u$. Then

$$(H \cdot A)(\omega, t) = 1_E(\omega)(A(\omega, t\wedge v) - A(\omega, t \wedge u)) \Rightarrow (H \cdot A)_t \in \mathcal{F}_t.$$

Let
$$\Pi = \{E \times (u,v] : E \in \mathcal{F}_u, u<v\} \subset \Omega \times [0,\infty).$$

Clearly, Π is a π-system (closed under intersection, nonempty), generating the predictable σ-algebra.

Let
$$\mathcal{V} = \{H : \Omega \times [0,\infty) \to \mathbb{R} : H \cdot A \text{ is adapted}\}.$$

Then: $1 \in \mathcal{V}$, $1_H \in \mathcal{V}$ for $H \in \Pi$ by the above, and if $0 \le H_n \in \mathcal{V}$ with $H_n \uparrow H$ then $H \in \mathcal{V}$ since measurability is closed under pointwise limits.

Thus $\mathcal{V}$ is a monotone class. By the monotone class theorem $\mathcal{V}$ contains all bounded predictable processes.

The general case follows by approximating H by bounded H^n with $|H^n| \le |H|$ and dominated convergence.

2.2. Local martingales

From now on, we assume that $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ satisfies the usual conditions ($\to$ A.P.):

- $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets;
- $(\mathcal{F}_t)$ is right-continuous, i.e., $\mathcal{F}_t = \mathcal{F}_{t+} = \bigcap_{s>t} \mathcal{F}_s$ for all $t \geq 0$.

Thm (Optional Stopping Theorem). Let X be a càdlàg adapted integrable process. Then the following are equivalent.

(i) X is a martingale: $\mathbb{E}(X_t | X_s) = X_s$ a.s. for all $t \geq s$

(ii) for stopping time S, T, with T bounded, one has $X_T \in L^1$,

$$\mathbb{E}(X_T | \mathcal{F}_S) = X_{S\wedge T} \text{ a.s.}$$

(iii) for all stopping times T, the process X^T where $X^T_t = X_{t\wedge T}$ is a martingale.

(iv) for all bounded stopping times T, one has $X_T \in L^1$ and

$$\mathbb{E}(X_T) = \mathbb{E}(X_0).$$

For X uniformly integrable, (ii) & (iv) hold for all stopping times T.

Defn. A càdlàg adapted process X is a local martingale if there are stopping times T_n s.t. $T_n \uparrow \infty$ as $n \to \infty$ and X^{T_n} is a martingale for every n. The sequence (T_n) is said to reduce X.

Example. (i) Every martingale is a local martingale. (Take $T_n = n$ and use the OST.)

(ii) Let (B_t) be a standard Brownian motion on $\mathbb{R}^3$. Then $(X_t)_{t\geq 1} = (1/|B_t|)_{t\geq 1}$ is a local martingale but not a martingale.

Proof. First, X cannot be a martingale since $(\to$ A.P.$)$

$$\sup_{t\geq 1} \mathbb{E}X_t^2 < \infty, \quad \mathbb{E}X_t \to 0.$$

To show that X is a local martingale nonetheless, recall that for $f \in C_b^2(\mathbb{R}^3)$,

$$f(B_t) - f(B_1) - \frac{1}{2}\int_1^t \Delta f(B_s)\, ds =: M_t^f$$

is a martingale. We would like to choose $f(x) = \frac{1}{|x|}$ so that $X_t = f(B_t)$, but f is not bounded at 0.

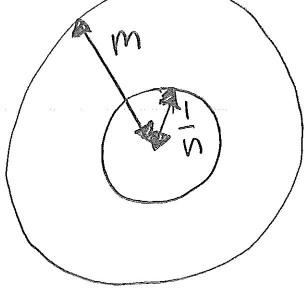
Choose $f_n \in C_b^2$ s.t. $f_n(x) = \frac{1}{|x|}$ for $|x| \geq \frac{1}{n}$. Let $T_n = \inf\{t \geq 1 : |B_t| < \frac{1}{n}\}$.

$$\Rightarrow \quad X_t^{T_n} - X_1^{T_n} = f_n(B_{t\wedge T_n}) - f_n(B_{1\wedge T_n}) = M^{f_n}_{t\wedge T_n}$$

since $\Delta f_n = \Delta \frac{1}{|x|}$ for $|x| \geq \frac{1}{n}$.

So X^{T_n} is a martingale. To conclude that X is a local martingale, it only remains to show that $T_n \to \infty$ a.s.

Let $S_m = \inf\{t \geq 1 : |B_t| > m\}$



$$\text{OST} \underset{\substack{\uparrow \\ X^{T_n} \text{ is a bounded martingale}}}{\Rightarrow} \mathbb{E}X_{T_n \wedge S_m} = \mathbb{E}X_1 < \infty.$$

But also

$$\mathbb{E} X_{T_n \wedge S_m} = n\,\mathbb{P}(T_n < S_m) + \frac{1}{m}\,\underbrace{\mathbb{P}(T_n \geq S_m)}_{1-\mathbb{P}(T_n < S_m)}$$

$$= \frac{1}{m} + \left(n - \frac{1}{m}\right)\mathbb{P}(T_n < S_m)$$

$$\Rightarrow \mathbb{P}(T_n < \infty) \leq \lim_{m\to\infty} \mathbb{P}(T_n < S_m) = \frac{\mathbb{E}X_1}{n}$$

$$\Rightarrow \mathbb{P}(\lim T_n < \infty) = 0, \text{ i.e., } T_n \to \infty \text{ a.s.}$$

The next proposition shows that X is also a supermartingale.

Prop. Let X be a local martingale and $X_t \geq 0$ for all t. Then X is a supermartingale.

Proof. Let (T_n) be a reducing sequence for X.

$$\Rightarrow \mathbb{E}(X_t \mid \mathcal{F}_s) = \mathbb{E}\left(\lim_{n\to\infty} X_{t\wedge T_n} \mid \mathcal{F}_s\right)$$

$$\leq \liminf_{n\to\infty} \mathbb{E}(X_{t\wedge T_n} \mid \mathcal{F}_s) \quad \text{by conditional Fatou a.s.}$$

$$= \liminf_{n\to\infty} X_{s\wedge T_n} = X_s \quad \text{a.s.}$$

*

<u>Lemma</u> (Example Sheet). Let $X \in L^1(\Omega, \mathcal{F}, \mathbb{P})$. Then the set

$$\mathcal{X} = \{\mathbb{E}(X \mid \mathcal{G}) : \mathcal{G} \subset \mathcal{F} \text{ is a sub-}\sigma\text{-algebra}\}$$

is uniformly integrable (UI), i.e.,

$$\sup_{Y \in \mathcal{X}} \mathbb{E}(|Y| 1_{|Y| > \lambda}) \to 0 \quad \text{as } \lambda \to \infty.$$

Recall (<u>Vitali's Theorem</u>). $X_n \to X$ in L^1 iff (X_n) is UI and $X_n \to X$ in probability.

<u>Prop</u>. The following are equivalent:
(a) X is a martingale
(b) X is a local martingale and for all $t \geq 0$ the set

$$\mathcal{X}_t = \{X_T : T \text{ is a stopping time}, T \leq t\} \text{ is UI.}$$

<u>Proof</u>. (a) $\Rightarrow$ (b). Let X be a martingale. Then by OST

$$X_T = \mathbb{E}(X_t \mid \mathcal{F}_T) \quad \text{for and stopping time } T \text{ with } T \leq t$$

By the previous lemma, $\mathcal{X}_t$ is UI.

(b) $\Rightarrow$ (a). Let X be a local martingale and let (T_n) be a reducing sequence for X. Assume that $\mathcal{X}_t$ is UI for all t. To show that X is a martingale, it suffices to show that $\mathbb{E}X_T = \mathbb{E}X_0$ for any bounded stopping time T (OST).

*

Let T be a bounded stopping time with $T \leq t$. Then

$$\mathbb{E} X_0 = \mathbb{E} X_0^{T_n} = \mathbb{E} X_T^{T_n} = \mathbb{E} X_{T \wedge T_n}.$$

Since $T \wedge T_n \to T$ a.s. and $\{X_{T \wedge T_n} : n \geq 0\}$ is UI, therefore $X_{T \wedge T_n} \to X_T$ in L^1.

$\Rightarrow \mathbb{E} X_0 = \mathbb{E} X_T$

Thus X is a martingale.

Cor. If there is $Z \in L^1$ s.t. $|X_t| \leq Z$ for all t, then X is a martingale. In particular, every bounded local martingale is a martingale.

Prop. Let X be a local martingale and suppose that there is $Z \in L^1$ s.t. $|X_t| \le Z$ for all t. Then X is a martingale. In particular, bounded local martingales are martingales.

Proof. Let (T_n) be a reducing sequence for X, and let S be a bounded stopping time. Then

$$\mathbb{E}X_0 = \mathbb{E}X_0^{T_n} = \mathbb{E}X_S^{T_n} = \mathbb{E}X_{T_n \wedge S}$$

Since $|X_{T_n \wedge S}| \le Z$, we have $X_{T_n \wedge S} \to X_S$ in L^1, so $\mathbb{E}X_0 = \mathbb{E}X_S$ and the claim follows from the OST.

Fact. Let X be a continuous adapted process with $X_0 = 0$. Then

$$S_n = \inf\{t \ge 0 : |X_t| = n\}$$

are stopping times for $(\mathcal{F}_t)$ and $S_n \uparrow \infty$ as $n \to \infty$.

Proof. S_n is a stopping time since

$$\{S_n \le t\} = \{\sup_{s \le t} |X_s| \ge n\} = \bigcap_{k \in \mathbb{N}} \bigcup_{\substack{s \in \mathbb{Q} \\ s \le t}} \{|X_s| > n - \tfrac{1}{k}\} \in \mathcal{F}_t .$$

$S_n \uparrow \infty$ since, for every $\omega \in \Omega$, $|X_s|$ is bounded on any bounded interval, by continuity.

Prop. Let X be a continuous local martingale with $X_0=0$. Then the sequence (S_n) of the previous fact reduces X.

Proof. Let (T_k) be a reducing sequence for X.
By OST, $X^{T_k \wedge S_n}$ is a martingale, so X^{S_n} is also a local martingale.
$|X^{S_n}| \leq n \Rightarrow X^{S_n}$ is a bounded local martingale
$\Rightarrow X^{S_n}$ is a martingale.

Thus S_n satisfies all conditions to be a reducing sequence for X.

Thm. Let X be a continuous local martingale with $X_0=0$. If X is also a finite variation process then $X_t=0 \ \forall t$ a.s.

Proof. Let
$$S_n = \inf\{t \geq 0 : \underbrace{\int_0^t |dX_s|}_{=V_t \text{ the total variation process of } X} = n\}.$$

Since S_n is a stopping time, X^{S_n} is a local martingale by OST. X^{S_n} is also bounded since
$$|X_t^{S_n}| \leq \int_0^{t \wedge S_n} |dX_s| \leq n.$$
$\Rightarrow X^{S_n}$ is a martingale.

Let $0=t_0<t_1<\cdots<t_k=t$ be a subdivision of $[0,t]$. Then

$$\mathbb{E}(X_t^{S_n})^2 = \sum_{i=1}^{k} \mathbb{E}\left((X_{t_i}^{S_n}-X_{t_{i-1}}^{S_n})^2\right) \quad \text{since } X^{S_n} \text{ is a martingale}$$

$$\leq \mathbb{E}\Bigg(\underbrace{\max_i |X_{t_i}^{S_n}-X_{t_{i-1}}^{S_n}|}_{\text{bounded (by } n)} \underbrace{\sum_{i=1}^{k} |X_{t_i}^{S_n}-X_{t_{i-1}}^{S_n}|}_{\leq \int_0^{t\wedge S_n} |dX_s| \leq n} \Bigg)$$

Take $\max_{0\leq i\leq k} |t_i-t_{i-1}|\to 0$. By continuity and DCT,

$$\mathbb{E}(X_t^{S_n})^2 = 0$$

$\Rightarrow X_{t\wedge S_n}=0$ a.s. $\forall t$ $\Rightarrow X_t=0$ a.s. $\forall t$

Since X is continuous, thus $X_t=0$ $\forall t$ a.s.

2.3. L^2 bounded martingales

<u>Defn</u>. Let

$$\mathcal{M}^2 = \{ X : \Omega \times [0,\infty) \to \mathbb{R} : X \text{ is a (càdlàg) martingale, } \sup_{t \geq 0} \mathbb{E} X_t^2 < \infty \} / \sim$$

$$\mathcal{M}_c^2 = \{ X \in \mathcal{M}^2 : X(\omega, \cdot) \text{ is continuous for every } \omega \in \Omega \} / \sim$$

where $\sim$ means that indistinguishable processes are identified. Moreover, set

$$\|X\|_{\mathcal{M}^2} = \left(\sup_t \mathbb{E} X_t^2 \right)^{1/2} = \left(\mathbb{E} X_\infty^2 \right)^{1/2}$$

Here recall that if $X \in \mathcal{M}^2$ then:

- $X_t \to X_\infty$ a.s. and in L^2.
- $(X_t^2)_{t \geq 0}$ is a submartingale, so $t \mapsto \mathbb{E} X_t^2$ is increasing,

$$\mathbb{E} X_\infty^2 = \sup_t \mathbb{E} X_t^2 .$$

- Doob's L^2 inequality:

$$\mathbb{E} \left(\sup_t X_t^2 \right) \leq 4 \mathbb{E} X_\infty^2 .$$

In particular, $\|X\|_{\mathcal{M}^2} = 0$ implies $X = 0$. This makes $\|\cdot\|_{\mathcal{M}^2}$ a norm (the other properties are clear). It comes from the inner product $\mathbb{E}(X_\infty Y_\infty)$ on $\mathcal{M}^2$.

<u>Prop</u>. $\mathcal{M}^2$ is a Hilbert space and $\mathcal{M}^2_c$ is a closed subspace.

<u>Proof</u>. We need to prove that $\mathcal{M}^2$ is complete. Thus let $(X^n) \subset \mathcal{M}^2$ be a Cauchy sequence:

$$\mathbb{E}(X^n_\infty - X^m_\infty)^2 \to 0 \quad \text{as } n,m\to\infty.$$

By passing to a subsequence, we may assume that

$$\mathbb{E}(X^n_\infty - X^{n-1}_\infty)^2 \leq 2^{-n}$$

and it suffices to show that the subsequence converges to conclude that the original sequence converges.

$$\mathbb{E}\Big(\sum_{n=1}^\infty \sup_{t\geq 0} |X^n_t - X^{n-1}_t|\Big) \overset{(CS)}{\leq} \sum_{n=1}^\infty \mathbb{E}\Big(\sup_{t\geq 0} |X^n_t - X^{n-1}_t|^2\Big)^{1/2}$$

$$\overset{(Doob)}{\leq} \sum_{n=1}^\infty 2\,\mathbb{E}\big(|X^n_\infty - X^{n-1}_\infty|^2\big)^{1/2} = \sum_n 2^{1-n/2} < \infty$$

$$\Rightarrow \sum_{n=1}^\infty \sup_{t\geq 0} |X^n_t - X^{n-1}_t| < \infty \text{ a.s.}$$

$\Rightarrow$ (X^n) is a Cauchy sequence in $D[0,\infty)$, $\|\cdot\|_\infty$ a.s.

$\Rightarrow$ $\|X^n - X\|_\infty \to 0$ a.s. for some $X \in D[0,\infty)$

Set $X=0$ outside the a.s. event. Then $X\in D[0,\infty)$ for every $\omega\in\Omega$.

Claim: $\mathbb{E}\Big(\sup_{t\geq 0} |X^n - X|^2\Big) \to 0$

$$\mathbb{E}\Big(\sup_t |X^n - X|^2\Big) = \mathbb{E}\Big(\lim_{m\to\infty} \sup_{t\geq 0} |X^n - X^m|^2\Big)$$

$$\overset{(Fatou)}{\leq} \liminf_{m\to\infty} \mathbb{E}\Big(\sup_{t\geq 0} |X^n - X^m|^2\Big) \overset{(Doob)}{\leq} 4 \liminf_{m\to\infty} \mathbb{E}(X^n_\infty - X^m_\infty)^2$$

$$\to 0$$

Claim: X is a martingale

$$\|\mathbb{E}(X_t|\mathcal{F}_s)-X_s\|_{L^2} \overset{(\Delta)}{\leq} \|\mathbb{E}(X_t-X_t^n|\mathcal{F}_s)\|_{L^2}+\|X_s^n-X_s\|_{L^2}$$

$$\overset{(\text{Jensen})}{\leq} \|X_t-X_t^n\|_{L^2}+\|X_s^n-X_s\|_{L^2}$$

$$\leq 2\,\mathbb{E}\Big(\sup_{t\geq 0}|X_t-X_t^n|^2\Big)^{1/2}\to 0$$

Thus $X\in\mathcal{M}^2$ and we have shown that $\mathcal{M}^2$ is complete.

Clearly, $\mathcal{M}^2_c$ is a subspace of $\mathcal{M}^2$. It is complete (and thus closed) by the same argument with $D[0,\infty)$ replaced by $C[0,\infty)$.

2.4. Quadratic variation

Defn. For a sequence of processes (X^n) and a process X, we say

$$X^n \to X \quad \text{ucp (uniformly on compact sets in probability)}$$

iff

$$\mathbb{P}\Big(\sup_{s\in[0,t]} |X^n_s - X_s| > \varepsilon\Big) \xrightarrow{n\to\infty} 0 \quad \forall t>0\ \forall \varepsilon>0 .$$

Thm. Let M be a continuous local martingale. Then there exists a unique (up to indistinguishability) continuous adapted increasing process $\langle M\rangle = (\langle M\rangle_t)$ s.t. $\langle M\rangle_0 = 0$ and $M^2 - \langle M\rangle$ is a continuous local martingale. Moreover with $0 = t_0^m < t_1^m < \cdots$ given by $t_i^m = 2^{-m} i$,

$$\langle M\rangle_t^{(m)} \xrightarrow{ucp} \langle M\rangle_t \quad \text{where } \langle M\rangle_t^{(m)} = \sum_{i=1}^{\lfloor 2^m t\rfloor} (M_{t_i} - M_{t_{i-1}})^2 .$$

(In fact, the convergence is true for all locally finite subdivisions of $[0,\infty)$ with $\max_i |t_i^m - t_{i-1}^m| \to 0$ as $m\to\infty$.)

Defn. $\langle M\rangle$ is the quadratic variation of M

Example. Let B be a standard Brownian motion. Then $B_t^2 - t$ is a martingale and thus $\langle B\rangle_t = t$.

By replacing M_t by $M_t - M_0$, it is sufficient to assume that $M_0 = 0$ from now on.

__Lemma__ (Uniqueness). There is at most one process $\langle M \rangle$ (up to indistinguishability) as asserted in the theorem.

__Proof.__ Suppose (A_t) and (B_t) both obey the conditions asserted for $\langle M \rangle$. Then

$$\underbrace{A_t - B_t}_{\substack{\text{finite variation} \\ \text{since difference} \\ \text{of continuous increasing processes}}} = \underbrace{(M_t^2 - B_t) - (M_t^2 - A_t)}_{\text{continuous local martingale}}$$

Hence $A - B = 0$ a.s.

__Lemma.__ (Stopping). Suppose M is a continuous local martingale for which $\langle M \rangle$ exists (as in the theorem). Let T be a stopping time. Then $\langle M^T \rangle$ exists and

$$\langle M^T \rangle_t = \langle M \rangle_{t \wedge T} \quad \text{(up to indistinguishability)}.$$

__Proof.__ Since $M_t^2 - \langle M \rangle_t$ is a continuous local martingale, so is $M_{t\wedge T}^2 - \langle M \rangle_{t \wedge T} = (M^T)_t^2 - \langle M \rangle_{t\wedge T}$. By uniqueness, thus $\langle M^T \rangle_t = \langle M \rangle_{t \wedge T}$ up to indistinguishability

Lemma (bounded case). The assertions of the theorem hold under the additional assumption that

$$|M_t| \le C \quad \text{for all } (\omega,t),$$

$$M_t = M_{t\wedge T}$$

for deterministic constants C and T.

Proof. Let

$$X_t^m = \sum_{i=1}^{\lfloor 2^m T\rfloor} M_{(i-1)2^{-m}}\left(M_{i2^{-m}\wedge t} - M_{(i-1)2^{-m}\wedge t}\right).$$

Then X^m is a bounded martingale.

Claim: $\langle M\rangle^{(m)}_{k2^{-m}} = M^2_{k2^{-m}} - 2X^m_{k2^{-m}}$

$$\langle M\rangle^{(m)}_{k2^{-m}} = \sum_{i=1}^{k} \underbrace{(M_{i2^{-m}} - M_{(i-1)2^{-m}})^2}$$

$$\underbrace{M_{i2^{-m}}}(M_{i2^{-m}} - M_{(i-1)2^{-m}}) - M_{(i-1)2^{-m}}(M_{i2^{-m}} - M_{(i-1)2^{-m}})$$

$$-M_{(i-1)2^{-m}} + (M_{i2^{-m}} + M_{(i-1)2^{-m}})$$

$$= \sum_{i=1}^{k}\left(M^2_{i2^{-m}} - M^2_{(i-1)2^{-m}}\right) - 2X^m_{k2^{-m}} = M^2_{k2^{-m}} - 2X^m_{k2^{-m}}$$

Also observe (for later) that $\langle M\rangle^{(m)}_t$ is increasing on $t \in \{2^{-m}i : i\in\mathbb{N}\}$.

Claim: $(X^m) \subset \mathcal{M}_c^2$ is Cauchy

It is clear that $X^m \in \mathcal{M}_c^2$ since M is a bounded martingale.

For $m' > m$,

$$X_\infty^{m'} - X_\infty^m = \sum_{i=1}^{\lfloor 2^{m'}T \rfloor} \left(M_{(i-1)2^{-m'}} - M_{\lfloor (i-1)2^{m-m'} \rfloor 2^{-m}} \right) \left(M_{i2^{-m'}} - M_{(i-1)2^{-m'}} \right)$$

$$\Rightarrow \mathbb{E}\left(X_\infty^{m'} - X_\infty^m \right)^2 \underset{\substack{\uparrow \\ \text{orthogonal increments}}}{=} \sum_{i=1}^{\lfloor 2^{m'}T \rfloor} \mathbb{E}\left(\left(M_{(i-1)2^{-m'}} - M_{\lfloor (i-1)2^{m-m'} \rfloor 2^{-m}} \right)^2 \left(M_{i2^{-m'}} - M_{(i-1)2^{-m'}} \right)^2 \right)$$

$$\leq \mathbb{E}\left(\sup_{|t-s| \leq 2^{-m}} |M_t - M_s|^2 \underbrace{\sum_{i=1}^{\lfloor 2^{m'}T \rfloor} \left(M_{i2^{-m'}} - M_{(i-1)2^{-m'}} \right)^2}_{\langle M \rangle_T^{(m')}} \right)$$

$$\leq \mathbb{E}\left(\sup_{|t-s| \leq 2^{-m}} |M_t - M_s|^4 \right)^{1/2} \mathbb{E}\left(\left(\langle M \rangle_T^{(m')} \right)^2 \right)^{1/2}$$

Claim: $\mathbb{E}\left(\sup_{|t-s| \leq 2^{-m}} |M_t - M_s|^4 \right) \to 0$ as $m \to \infty$

Indeed, $|M_t - M_s|^4 \leq (2C)^4$

$$\sup_{|t-s| \leq 2^{-m}} |M_t - M_s| \to 0 \text{ by uniform continuity}$$

$$\Rightarrow \mathbb{E}\left(\sup_{|t-s| \leq 2^{-m}} |M_t - M_s|^4 \right) \to 0 \text{ by DCT}$$

Claim: $\mathbb{E}\left(\left(\langle M\rangle_T^{(m)}\right)^2\right) \leq 48C^4$

$$\mathbb{E}\left(\left(\sum_{i=1}^{n}(M_{i2^{-m}}-M_{(i-1)2^{-m}})^2\right)^2\right) = \sum_{i=1}^{n}\mathbb{E}(M_{i2^{-m}}-M_{(i-1)2^{-m}})^4$$

$$+2\sum_{i=1}^{n}\underbrace{\mathbb{E}\left((M_{i2^{-m}}-M_{(i-1)2^{-m}})^2\sum_{k=i+1}^{n}(M_{k2^{-m}}-M_{(k-1)2^{-m}})^2\right)}_{\mathbb{E}(M_{i2^{-m}}-M_{(i-1)2^{-m}})^2(M_{n2^{-m}}-M_{i2^{-m}})^2 \text{ by orthogonality}}$$

$$\leq (4+8)C^2\sum_{i=1}^{n}\mathbb{E}(M_{i2^{-m}}-M_{(i-1)2^{-m}})^2$$

$$= 12C^2\,\mathbb{E}(M_{n2^{-m}}-M_0)^2$$

by orthogonality

$$\leq 48C^4.$$

$\Rightarrow \mathbb{E}(X_\infty^{m'}-X_\infty^{m})^2 \to 0$ as $m,m'\to\infty$.

$\Rightarrow (X^m)$ is Cauchy in $\mathcal{M}_c^2$

$\Rightarrow$ There is $X\in\mathcal{M}_c^2$ s.t. $X^m\to X$ in $\mathcal{M}_c^2$.

Since $X^m\to X$ in $\mathcal{M}_c^2$, in particular $\|\sup_t|X_t^m-X_t|\|_{L^2}\to 0$.

$\Rightarrow \sup_t|X_t^m-X_t|\to 0$ a.s. for some subsequence

Let $N\subset\Omega$ be the event with $\mathbb{P}(N)=0$ on which this convergence fails.

Set
$$\langle M\rangle_t = \begin{cases} M_t^2 - 2X_t & \text{for } \omega\in\Omega\setminus N \\ 0 & \text{for } \omega\in N \end{cases}$$

Then $\langle M\rangle$ is continuous and adapted since M and X are.

$M^2 - \langle M\rangle = 2X$ is a martingale since X is.

$\langle M\rangle$ is increasing since $M^2 - 2X^m$ is increasing on the set of times $i2^{-m}$, $i\in\mathbb{Z}_+$, and the convergence $M^2 - 2X^m \longrightarrow \langle M\rangle$ is uniform.

Claim: $\langle M\rangle^{(m)} \longrightarrow \langle M\rangle$ ucp

Recall that $\langle M\rangle_t^{(m)} = \langle M\rangle^{(m)}_{\lfloor 2^m t\rfloor 2^{-m}} = M^2_{\lfloor 2^m t\rfloor 2^{-m}} - 2X^m_{\lfloor 2^m t\rfloor 2^{-m}}$

$\sup_t |X_t^m - X_t| \to 0$ in L^2 and thus in prob.

$$\Rightarrow \sup_t |\langle M\rangle_t - \langle M\rangle_t^{(m)}| \le \underbrace{\sup_t |M^2_{\lfloor 2^m t\rfloor 2^{-m}} - M_t^2|}_{\to 0 \text{ a.s. by uniform continuity}}$$

$$+ \underbrace{\sup_t |X_{\lfloor 2^m t\rfloor 2^{-m}} - X_t|}_{\to 0 \text{ a.s. by uniform continuity}}$$

$$+ \underbrace{\sup_t |X^m_{\lfloor 2^m t\rfloor 2^{-m}} - X_{\lfloor 2^m t\rfloor 2^{-m}}|}_{\to 0 \text{ in prob. as above}}.$$

<u>Proof of theorem</u> (general case). Let M be a continuous local martingale. Let

$$T_n = \inf\{t \geq 0 : |M_t| > n\}, \quad S_n = T_n \wedge n.$$

Then $S_n \uparrow \infty$ and M^{S_n} is a bounded martingale s.t. $(M^{S_n})_{t \wedge n} = (M^{S_n})_t$. By the previous lemma, $\langle M^{S_n} \rangle$ exists. By uniqueness,

$$\langle M^{S_n} \rangle_t = \langle M^{S_{n+1}} \rangle_{t \wedge S_n}$$

Thus there is a process $\langle M \rangle$ s.t. $\langle M \rangle_{t \wedge S_n}$ and $\langle M^{S_n} \rangle$ are indistinguishable for all $n \in \mathbb{N}$.

Clearly, $\langle M \rangle$ is increasing since the $\langle M^{S_n} \rangle$ are; and $(M^2 - \langle M \rangle)^{S_n}$ is a martingale for all n, $S_n \uparrow \infty$, so $M^2 - \langle M \rangle$ is a local martingale.

Claim: $\langle M \rangle^{(m)} \longrightarrow \langle M \rangle$ ucp

By the previous lemma, $\langle M^{S_n} \rangle^{(m)} \longrightarrow \langle M^{S_n} \rangle$ for every n.

$$\Rightarrow \mathbb{P}\left(\sup_{t \leq T} |\langle M \rangle_t^{(m)} - \langle M \rangle_t| > \varepsilon \right) \leq \mathbb{P}(S_n < T) + \underbrace{\mathbb{P}\left(\sup_{t \leq T} |\langle M^{S_n} \rangle_t^{(m)} - \langle M^{S_n} \rangle_t| > \varepsilon \right)}_{\to 0 \text{ for any } n}$$

$\to 0$ since $\mathbb{P}(S_n < T) \to 0$ as $n \to \infty$.

Fact. Let M be a continuous local martingale with $M_0 = 0$. Then $M = 0$ iff $\langle M \rangle = 0$.

Proof. If $\langle M \rangle = 0$ then M^2 is a continuous local martingale and positive, so a supermartingale. Thus $\mathbb{E} M_t^2 \leq \mathbb{E} M_0^2 = 0$ for all t.

Prop. Let M be a cont. loc. mart. with $M_0 = 0$. Then $M \in \mathcal{M}_c^2$ iff $\mathbb{E}\langle M \rangle_\infty < \infty$ and then $M^2 - \langle M \rangle$ is a UI mart. and

$$\|M\|_{\mathcal{M}^2} = \left(\mathbb{E}\langle M \rangle_\infty\right)^{1/2}.$$

Proof. Assume first that $M \in \mathcal{M}_c^2$ and $\mathbb{E}\langle M \rangle_\infty < \infty$. Then

$$|M_t^2 - \langle M \rangle_t| \leq \underbrace{\sup_{t \geq 0} M_t^2 + \langle M \rangle_\infty}_{= Z \, \in L^1 \text{ by Doob's } L^2 \text{ inequality}} \quad \text{for all } t$$

Thus $M^2 - \langle M \rangle$ is a continuous local martigale bounded by $Z \in L^1$. In particular, $M^2 - \langle M \rangle$ is uniformly integrable and we have also seen that $M^2 - \langle M \rangle$ is a martingale.

Claim: $\langle M \rangle_\infty \in L^1$ if $M \in \mathcal{M}_c^2$.

Let $S_n = \inf\{t \geq 0 : \langle M \rangle_t \geq n\}$. Then $S_n \uparrow \infty$, S_n is a stopping time, and $\langle M \rangle_{t \wedge S_n} \leq n$.

$$\Rightarrow M^2_{t\wedge S_n} - \langle M\rangle_{t\wedge S_n} \leq n + \underbrace{\sup_{t\geq 0} M_t^2}_{\in L^1}$$

$\Rightarrow M^2_{t\wedge S_n} - \langle M\rangle_{t\wedge S_n}$ is a (true) martingale

$$\Rightarrow \mathbb{E} M^2_{t\wedge S_n} = \mathbb{E}\langle M\rangle_{t\wedge S_n}$$

Take $t\to\infty$: $\mathbb{E}M^2_{t\wedge S_n} \to \mathbb{E}M^2_{S_n}$ by DCT $\left(\mathbb{E}\sup_t \mathbb{E}M^2_{t\wedge S_n} < \infty\right)$

$\mathbb{E}\langle M\rangle_{t\wedge S_n} \to \mathbb{E}\langle M\rangle_{S_n}$ by DCT $\left(\sup_t \langle M\rangle_{t\wedge S_n} \leq n\right)$

Take $n\to\infty$: $\mathbb{E}M^2_{S_n} \to \mathbb{E}M^2_\infty$ by DCT $\left(\mathbb{E}\sup_t M_t^2 < \infty\right)$

$\mathbb{E}\langle M\rangle_{S_n} \to \mathbb{E}\langle M\rangle_\infty$ by monotone convergence

$$\Rightarrow \mathbb{E}\langle M\rangle_\infty = \mathbb{E}M^2_\infty < \infty.$$

Claim: $M\in \mathcal{M}_c^2$ if $\mathbb{E}\langle M\rangle_\infty < \infty$.

The proof is similar using $S_n = \inf\{t\geq 0;\ |M_t|\geq n\}$. Again, $M^2_{t\wedge S_n} - \langle M\rangle_{t\wedge S_n}$ is a (true) martingale.

By Fatou,

$$\mathbb{E}M_t^2 \leq \liminf_{n\to\infty} \mathbb{E}M^2_{t\wedge S_n} = \liminf_{n\to\infty} \mathbb{E}\langle M\rangle_{t\wedge S_n} \leq \mathbb{E}\langle M\rangle_\infty.$$

2.5. Covariation

<u>Defn.</u> For M and N continuous local martingales, define

$$\langle M,N\rangle = \frac{1}{4}\big(\langle M+N\rangle - \langle M-N\rangle\big)$$

The process $\langle M,N\rangle = (\langle M,N\rangle_t)_t$ is the <u>covariation</u> or <u>bracket</u> of M and N.

<u>Prop.</u>

(i) $\langle M,N\rangle$ is the unique (up to indistinguishability) finite variation process s.t. $MN - \langle M,N\rangle$ is a continuous local martingale.

(ii) The mapping $(M,N) \mapsto \langle M,N\rangle$ is bilinear and symmetric.

(iii) $$\langle M,N\rangle_t^{(m)} \xrightarrow{ucp} \langle M,N\rangle_t \quad \text{where } \langle M,N\rangle_t^{(m)} = \sum_{i=1}^{\lfloor 2^m t\rfloor} \big(M_{i2^{-m}} - M_{(i-1)2^{-m}}\big) \times \big(N_{i2^{-m}} - N_{(i-1)2^{-m}}\big)$$

(iv) For every stopping time T,
$$\langle M^T, N^T\rangle_t = \langle M^T, N\rangle_t = \langle M,N\rangle_{t\wedge T}.$$

(v) If $M,N \in \mathcal{M}_c^2$ then $M_t N_t - \langle M,N\rangle_t$ is a UI martingale and
$$(M-M_0, N-N_0)_{\mathcal{M}^2} = \mathbb{E}\langle M,N\rangle_\infty$$

<u>Proof.</u> Exactly as for $M=N$.

Example. Let B and B' be independent Brownian motions (adapted w.r.t. the same filtration). Then BB' is a martingale (by independence), so $\langle B,B'\rangle = 0$. Let $B'' = \rho B + \sqrt{1-\rho^2}\,B'$ for some $\rho \in [0,1]$. Then B'' is also a BM, and by bilinearity

$$\langle B, B''\rangle = \rho t.$$

Prop. (Kunita–Watanabe inequality). Let M and N be continuous local martingales, and let H and K be measurable processes (not necessarily adapted). Then a.s.

$$\int_0^\infty |H_s|\,|K_s|\,|d\langle M,N\rangle_s| \le \left(\int_0^\infty |H_s|^2\,|d\langle M\rangle_s|\right)^{1/2}\left(\int_0^\infty |K_s|^2\,|d\langle N\rangle_s|\right)^{1/2}. \qquad \text{(KW)}$$

Idea: approximate and apply Cauchy–Schwarz.

Proof. Write $\langle M,N\rangle_s^t = \langle M,N\rangle_t - \langle M,N\rangle_s$.

Claim: For all $0 \le s < t$,

$$|\langle M,N\rangle_s^t| \le \sqrt{\langle M,M\rangle_s^t}\,\sqrt{\langle N,N\rangle_s^t}. \qquad (*)$$

By continuity, we can assume that s and t are dyadic rationals. Then indeed

$$|\langle M,N\rangle_s^t| \overset{\text{(Prop.)}}{=} \lim_{n\to\infty}\left|\sum_{i=2^n s+1}^{2^n t}(M_{2^{-n}i} - M_{2^{-n}(i-1)})(N_{2^{-n}i} - N_{2^{-n}(i-1)})\right|$$

$$\overset{\text{(CS)}}{\le} \lim_{n\to\infty}\left(\sum_{i=2^n s+1}^{2^n t}(M_{2^{-n}i} - M_{2^{-n}(i-1)})^2\right)^{1/2}\left(\sum_{i=2^n s+1}^{2^n t}(N_{2^{-n}i} - N_{2^{-n}(i-1)})^2\right)^{1/2}$$

$\Rightarrow |\langle M,N\rangle_s^t| \leq (\langle M,M\rangle_s^t)^{1/2} (\langle N,N\rangle_s^t)^{1/2}$

Now fix an event s.t. (*) holds for all $s<t$ rational and thus by continuity also for all $s<t$ irrational).

Claim: $\int_s^t |d\langle M,N\rangle_u| \leq \sqrt{\langle M,M\rangle_s^t} \sqrt{\langle N,N\rangle_s^t}$.

Indeed, for any subdivision $s=t_0<t_1<\dots<t_n=t$,

$$\sum_{i=1}^n |\langle M,N\rangle_{t_{i-1}}^{t_i}| \leq \sum_{i=1}^n \sqrt{\langle M,M\rangle_{t_{i-1}}^{t_i}} \sqrt{\langle N,N\rangle_{t_{i-1}}^{t_i}}$$
$$\overset{(CS)}{\leq} \left(\sum_{i=1}^n \langle M,M\rangle_{t_{i-1}}^{t_i}\right)^{1/2} \left(\sum_{i=1}^n \langle N,N\rangle_{t_{i-1}}^{t_i}\right)^{1/2}$$
$$= (\langle M,M\rangle_s^t)^{1/2} (\langle N,N\rangle_s^t)^{1/2}.$$

The claim follows by taking the sup over all subdivisions.

Claim: For all bounded Borel sets $B \subset [0,\infty)$,

$$\int_B |d\langle M,N\rangle_s| \leq \sqrt{\int_B |d\langle M,M\rangle_u|} \sqrt{\int_B |d\langle N,N\rangle_u|}$$

For B a finite union of intervals, this follows from the Cauchy-Schwarz inequality as above.

Exercise: extend to all bounded Borel sets by a monotone class argument.

Claim: (KW) holds for $H = \sum h_e 1_{B_e}$, $K = \sum k_e 1_{B_e}$ where the B_e are bounded Borel sets.

$$\int |H_s K_s| \, |d\langle M,N\rangle_s| = \sum_e |h_e k_e| \int_{B_e} |d\langle M,N\rangle_s|$$

$$\leq \sum_e |h_e k_e| \left(\int_{B_e} |d\langle M\rangle_s| \right)^{1/2} \left(\int_{B_e} |d\langle N\rangle_s| \right)^{1/2}$$

$$\overset{(CS)}{\leq} \left(\sum_e h_e^2 \int_{B_e} |d\langle M\rangle_s| \right)^{1/2} \left(\sum_e k_e^2 \int_{B_e} |d\langle N\rangle_s| \right)^{1/2}$$

$$= \left(\int H_s^2 \, |d\langle M\rangle_s| \right)^{1/2} \left(\int K_s^2 \, |d\langle N\rangle_s| \right)^{1/2}$$

Finally, approximate general H, K by simple functions.

2.6. Semimartingales

Defn. A (continuous) adapted process X is a continuous semimartingale if

$$X = X_0 + M + A$$

with $X_0 \in \mathcal{F}_0$, M a (continuous) local martingale with $M_0 = 0$, and A a (continuous) bounded variation process with $A_0 = 0$.

Rk. The decomposition is unique (up to indistinguishability).

Defn. Let $X = X_0 + M + A$ and $X' = X_0' + M' + A'$ be continuous semimartingales. Set

$$\langle X \rangle = \langle M \rangle, \qquad \langle X, X' \rangle = \langle M, M' \rangle.$$

Prop.

$$\langle X, Y \rangle_t^{(m)} = \sum_{i=1}^{\lfloor 2^m t \rfloor} (X_{i2^{-m}} - X_{(i-1)2^{-m}})(Y_{i2^{-m}} - Y_{(i-1)2^{-m}}) \xrightarrow{\text{ucp}} \langle X, Y \rangle_t$$

Proof. Exercise.

3. The Itô integral

3.1. Simple processes

Defn. The space of simple processes $\mathcal{E}$ consists of $H : \Omega \times [0,\infty) \to \mathbb{R}$ that can be written as

$$H(\omega, t) = \sum_{i=1}^{n} H_{i-1}(\omega) \mathbb{1}_{(t_{i-1}, t_i]}(t)$$

for $0 \le t_0 < \dots < t_n$ and bounded random variables $H_{i-1} \in \mathcal{F}_{t_{i-1}}$.

Defn. For $M \in \mathcal{M}^2_c$ and $H \in \mathcal{E}$, set (Itô integral for simple processes)

$$(H \cdot M)_t = \sum_{i=1}^{n} H_{i-1} (M_{t_i \wedge t} - M_{t_{i-1} \wedge t}).$$

We also write $\int_0^t H_s \, dM_s = (H \cdot M)_t$.

Prop. Let $M \in \mathcal{M}^2_c$ and $H \in \mathcal{E}$. Then $H \cdot M \in \mathcal{M}^2_c$ and

$$\|H \cdot M\|_{\mathcal{M}^2} = \mathbb{E}\left(\int_0^\infty H_s^2 \, d\langle M \rangle_s \right). \qquad (*)$$

(Itô isometry for simple processes).

Proof. Claim: $H \cdot M$ is a martingale in $\mathcal{M}^2_c$.

Let $X^i_t = H_{i-1}(M_{t_i \wedge t} - M_{t_{i-1} \wedge t})$. Then $H \cdot M = \sum_{i=1}^{n} X^i$ and it suffices to show that $X^i \in \mathcal{M}^2_c$.

Indeed,

for $s \geq t_{i-1}$: $\mathbb{E}(X_t^i \mid \mathcal{F}_s) = H_{i-1}\Big(\underbrace{\mathbb{E}(M_{t\wedge t_i} \mid \mathcal{F}_s)}_{M_{s\wedge t_i}} - M_{t_{i-1}}\Big) = X_s^i$

for $s < t_{i-1}$: $\mathbb{E}(X_t^i \mid \mathcal{F}_s) = \mathbb{E}\Big(H_{i-1}\,\underbrace{\mathbb{E}(M_{t\wedge t_i} - M_{t\wedge t_{i-1}} \mid \mathcal{F}_{t_{i-1}})}_{0} \mid \mathcal{F}_s\Big) = 0 = X_s^i$

Also, $\|X^i\|_{\mathcal{M}^2} \leq 2\|H\|_\infty \|M\|_{\mathcal{M}^2} < \infty$ so $X^i \in \mathcal{M}_c^2$.

Claim: (*) holds

$$\mathbb{E}\, X_\infty^i X_\infty^j = \mathbb{E}\Big(H_{i-1}(M_{t_i} - M_{t_{i-1}})\, H_{j-1}\, \underbrace{\mathbb{E}(M_{t_j} - M_{t_{j-1}} \mid \mathcal{F}_{t_{j-1}})}_{0}\Big) = 0 \quad (j > i)$$

so X^i and X^j are orthogonal

$$\Rightarrow \|H\cdot M\|_{\mathcal{M}^2}^2 = \sum_{i=1}^n \|X^i\|_{\mathcal{M}^2}^2 = \sum_{i=1}^n \mathbb{E}(X_\infty^i)^2$$

$$\mathbb{E}(X_\infty^i)^2 = \mathbb{E}\Big(H_{i-1}^2\, \underbrace{\mathbb{E}\big((M_{t_i} - M_{t_{i-1}})^2 \mid \mathcal{F}_{t_{i-1}}\big)}\Big)$$

$$\mathbb{E}(M_{t_i}^2 + M_{t_{i-1}}^2 - 2M_{t_i}M_{t_{i-1}} \mid \mathcal{F}_{t_{i-1}})$$

$$= \mathbb{E}(M_{t_i}^2 - M_{t_{i-1}}^2 \mid \mathcal{F}_{t_{i-1}})$$

$$= \mathbb{E}(\langle M\rangle_{t_i} - \langle M\rangle_{t_{i-1}} \mid \mathcal{F}_{t_{i-1}})$$

$$\Rightarrow \mathbb{E}(X_\infty^i)^2 = \mathbb{E}\big(H_{i-1}^2(\langle M\rangle_{t_i} - \langle M\rangle_{t_{i-1}})\big) = \mathbb{E}\int_{t_{i-1}}^{t_i} H_s^2\, d\langle M\rangle_s .$$

$$\Rightarrow \|H\cdot M\|_{\mathcal{M}^2}^2 = \mathbb{E}\int_0^\infty H_s^2\, d\langle M\rangle_s .$$

Prop. Let $M \in \mathcal{M}_c^2$ and let $H \in \mathcal{E}$. Then

$$\langle H \cdot M, N \rangle = H \cdot \langle M, N \rangle \qquad \forall N \in \mathcal{M}_c^2,$$

i.e., $\left\langle \int_0^{\cdot} H_s \, dM_s , N \right\rangle = \int_0^{\cdot} H_s \, d\langle M, N \rangle_s$.

Proof. Let $H \cdot M = \sum_{i=1}^{n} X^i$ as in the previous proof.

$$\Rightarrow \langle X^i, N \rangle_t = H_{i-1} \langle M_{t_i \wedge \cdot} - M_{t_{i-1} \wedge \cdot}, N \rangle_t$$

$$= H_{i-1} \left(\langle M, N \rangle_{t_i \wedge t} - \langle M, N \rangle_{t_{i-1} \wedge t} \right)$$

$$\Rightarrow \langle H \cdot M, N \rangle_t = \int_0^t H_s \, d\langle M, N \rangle_s = \left(H \cdot \langle M, N \rangle \right)_t .$$

3.2. Itô isometry

Defn. For $M \in \mathcal{M}_c^2$, define $L^2(M)$ to be the space of (equivalence classes) of predictable $H : \Omega \times [0,\infty) \to \mathbb{R}$ s.t.

$$\|H\|_{L^2(M)} = \|H\|_M = \left(\mathbb{E}\left(\int_0^\infty H_s^2 \, d\langle M\rangle_s\right)\right)^{1/2} < \infty.$$

For $H, K \in L^2(M)$, set

$$(H,K)_{L^2(M)} = (H,K)_M = \mathbb{E}\left(\int_0^\infty H_s K_s \, d\langle M\rangle_s\right).$$

Fact. $L^2(M) = L^2(\Omega \times [0,\infty), \mathcal{P}, dP \, d\langle M\rangle)$ is a Hilbert space.

Prop. Let $M \in \mathcal{M}_c^2$. Then $\mathcal{E}$ is dense in $L^2(M)$.

Since $L^2(M)$ is a Hilbert space (complete!) it suffices to show that if $(K,H)_M = 0 \ \forall H \in \mathcal{E}$ then $K = 0$.

So assume that $(K,H)_M = 0 \ \forall H \in \mathcal{E}$ and set

$$X_t = \int_0^t K_s \, d\langle M\rangle_s .$$

X is a well-defined finite variation process with $X_t \in L^1$ for all t since

$$\mathbb{E} \int_0^t |K_s| \, d\langle M_s\rangle \overset{(CS)}{\leq} \left(\mathbb{E}\left(\int_0^t |K_s|^2 \, d\langle M\rangle_s\right)\right)^{1/2} \left(\mathbb{E} \langle M\rangle_\infty\right)^{1/2} < \infty$$

where we used that $K \in L^2(M)$ and $M \in \mathcal{M}_c^2$.

Claim: X is a continuous martingale

Let $0 \le s < t$, $F \in \mathcal{F}_s$, $H = F \mathbb{1}_{(s,t]} \in \mathcal{E}$, F bounded.

By assumption, then

$$0 = (K, H)_M = \mathbb{E}\left(F \int_s^t K_u \, d\langle M \rangle_u\right)$$

$$= \mathbb{E}\left(F(X_t - X_s)\right) \quad \forall s < t, \; F \in \mathcal{F}_s \text{ bounded}$$

$\Rightarrow \mathbb{E}(X_t \mid \mathcal{F}_s) = X_s$ a.s., i.e. X is a continuous martingale

So X is a finite variation process and a continuous martingale, hence $X = 0$.

$\Rightarrow K_u = 0$ for $d\langle M \rangle$-a.e. u, a.s.

$\Rightarrow K = 0$ in $L^2(M)$.

<u>Thm</u>. Let $M \in \mathcal{M}_c^2$. Then

(i) The map $H \in \mathcal{E} \mapsto H \cdot M \in \mathcal{M}_c^2$ extends uniquely to an isometry $L^2(M) \to \mathcal{M}_c^2$, the <u>Itô isometry</u>.

(ii) $H \cdot M$ is the unique martingale in $\mathcal{M}_c^2$ s.t.

$$\langle \underbrace{H \cdot M}_{\text{Itô integral}}, N \rangle = \underbrace{H \cdot \langle M, N \rangle}_{\text{finite variation integral}} \quad \forall N \in \mathcal{M}_c^2$$

i.e. $\left\langle \int_0^{\cdot} H_s \, dM_s, N_{\cdot} \right\rangle = \int_0^{\cdot} H_s \, d\langle M, N \rangle_s$

Defn. $H \cdot M$ is the Itô integral of H w.r.t. M and we write

$$(H \cdot M)_t = \int_0^t H_s \, dM_s.$$

Proof. (i) For $H \in \mathcal{E}$, we have already seen that

$$\|H \cdot M\|_{M^2}^2 = \mathbb{E}\left(\int_0^\infty H_s^2 \, d\langle M \rangle_s\right) = \|H\|_{L^2(M)}^2 .$$

Since $\mathcal{E} \subset L^2(M)$ is dense and $\mathcal{M}_c^2$ is a Hilbert space, it follows that the map $H \mapsto H \cdot M$ extends uniquely to all of $L^2(M)$ and the extension is also an isometry.

(ii) Again, we have already seen that $\langle H \cdot M, N \rangle = H \cdot \langle M, N \rangle$ holds for $H \in \mathcal{E}$. Given $H \in L^2(M)$, choose $(H^n) \subset \mathcal{E}$ s.t. $H^n \to H$ in $L^2(M)$. Then $H^n \cdot M \to H \cdot M$ by (i).
We will justify

$$\begin{aligned} \langle H \cdot M, N \rangle_\infty &\overset{(+)}{=} \lim_{n \to \infty} \langle H^n \cdot M, N \rangle_\infty \quad \text{in } L^1 \\ &= \lim_{n \to \infty} (H^n \cdot \langle M, N \rangle)_\infty \\ &\overset{(+)}{=} (H \cdot \langle M, N \rangle)_\infty \quad \text{in } L^1 \end{aligned}$$

Here (†) holds by the Kunita-Watanabe inequality:

$$\mathbb{E}\,|\underbrace{\langle H\cdot M - H^n\cdot M, N\rangle_\infty}_{=\int_0^\infty 1\cdot 1\, d\langle H\cdot M - H^n\cdot M, N\rangle_s}| \le \left(\mathbb{E}\langle H\cdot M - H^n\cdot M\rangle_\infty\right)^{1/2}\left(\mathbb{E}\langle N\rangle_\infty\right)^{1/2}$$

$$= \underbrace{\|H\cdot M - H^n\cdot M\|_{\mathcal{M}^2}}_{\|H-H^n\|_{L^2(M)}\to 0}\|N\|_{\mathcal{M}^2}$$

$$\mathbb{E}\left((H-H^n)\cdot\langle M,N\rangle\right)_\infty \le \underbrace{\|H-H^n\|_{L^2(M)}}_{\to 0}\|N\|_{\mathcal{M}^2}$$

Thus $\langle H\cdot M, N\rangle_\infty = (H\cdot\langle M,N\rangle)_\infty$.

Replacing N by the stopped martingale N^t gives

$$\langle H\cdot M, N\rangle_t = \langle H\cdot M, N^t\rangle_\infty = (H\cdot\langle M, N^t\rangle)_\infty = (H\cdot\langle M,N\rangle)_t.$$

Uniqueness: assume that $X\in\mathcal{M}_c^2$ also satisfies

$$\langle X,N\rangle = H\cdot\langle M,N\rangle \quad \forall N\in\mathcal{M}_c^2.$$

$\Rightarrow \langle H\cdot M - X, N\rangle = 0 \quad \forall N\in\mathcal{M}_c^2$

$\Rightarrow \langle H\cdot M - X, H\cdot M - X\rangle = 0$

$\Rightarrow \|H\cdot M - X\|_{\mathcal{M}^2}^2 = 0$

$\Rightarrow X = H\cdot M.$

Cor. If T is a stopping time then

$$(\mathbb{1}_{[0,T]}H)\cdot M = (H\cdot M)^T = H\cdot M^T$$

Proof. For any $N \in \mathcal{M}^2$,

$$\langle (H\cdot M)^T, N\rangle_t = \langle H\cdot M, N\rangle_{t\wedge T} \overset{(ii)}{=} (H\cdot\langle M,N\rangle)_{t\wedge T}$$

$$= (H\mathbb{1}_{[0,T]}\cdot\langle M,N\rangle)_t$$

$\Rightarrow (H\cdot M)^T = H\mathbb{1}_{[0,T]}\cdot M$ by (ii)

Similarly,

$$\langle H\cdot M^T, N\rangle_t \overset{(ii)}{=} (H\cdot\langle M^T,N\rangle)_t = (H\cdot\langle M,N\rangle)_{t\wedge T}$$

$$= (H\mathbb{1}_{[0,T]}\cdot\langle M,N\rangle)_t$$

$\Rightarrow H\cdot M^T = H\mathbb{1}_{[0,T]}\cdot M$ by (ii).

Cor.

$$\langle H\cdot M, K\cdot N\rangle = (HK)\cdot\langle M,N\rangle$$

i.e. $\left\langle \int_0^\cdot H_s\,dM_s, \int_0^\cdot K_s\,dN_s\right\rangle_t = \int_0^t H_sK_s\,d\langle M,N\rangle_s$.

associativity of f.v. integral:
$(hk)df = h(k\,df) = hd(k\cdot f)$

Proof. $\langle H\cdot M, K\cdot N\rangle = H\cdot\langle M, K\cdot N\rangle = H\cdot(K\cdot\langle M,N\rangle) = (HK)\cdot\langle M,N\rangle$

Cor.

$$\mathbb{E}\left(\int_0^t H_s\,dM_s\right)=0,\quad \mathbb{E}\left(\int_0^t H_s\,dM_s \,\Big|\, \mathcal{F}_u\right)=\int_0^u H_s\,dM_s$$

$$\mathbb{E}\left(\int_0^t H_s\,dM_s \int_0^t K_s\,dN_s\right)=\mathbb{E}\int_0^t H_sK_s\,d\langle M,N\rangle_s$$

Proof. $H\bullet M$ and $(H\bullet M)(K\bullet N)-\langle H\bullet M, K\bullet N\rangle$ are martingales starting at 0.

Cor. Let $H\in L^2(M)$. Then $KH\in L^2(M)$ iff $K\in L^2(H\bullet M)$ and then

$$(KH)\bullet M = K\bullet(H\bullet M).$$

Proof Since $H^2\bullet\langle M\rangle = \langle H\bullet M\rangle$,

$$\mathbb{E}\left(\int_0^\infty K_s^2\underbrace{H_s^2\,d\langle M\rangle_s}_{d(H^2\bullet\langle M\rangle)_s}\right)=\mathbb{E}\left(\int_0^\infty K_s^2\,d\langle H\bullet M\rangle_s\right)$$

so $KH\in L^2(M)\Leftrightarrow K\in L^2(H\bullet M)$.

For $N\in \mathcal{M}_c^2$,

$$\langle (KH)\bullet M, N\rangle_t=(KH\bullet\langle M,N\rangle)_t=\int_0^t K_s\underbrace{H_s\,d\langle M,N\rangle_s}_{d\int_0^\cdot H_u\,d\langle M,N\rangle_u=d(H\bullet\langle M,N\rangle)}=(K\bullet(H\bullet\langle M,N\rangle))_t$$

$$\langle K\bullet(H\bullet M),N\rangle_t=(K\bullet\langle H\bullet M,N\rangle)_t=(K\bullet(H\bullet\langle M,N\rangle))_t$$

$\Rightarrow (KH)\bullet M = K\bullet(H\bullet M)$ by uniqueness.

3.3. Extension to local martingales

Defn. Let $L^2_{loc}(M)$ be the space of (equiv. classes) of predictable H s.t. a.s.

$$\int_0^t H_s^2 \, d\langle M\rangle_s < \infty \quad \text{for all } t.$$

Thm. Let M be a continuous local martingale.

(i) For every $H \in L^2_{loc}(M)$ there is a unique (up to indist.) continuous local martingale $H \bullet M$ with $(H \bullet M)_0 = 0$ s.t.

$$\langle H \bullet M, N\rangle = H \bullet \langle M, N\rangle \quad \forall N \text{ cont. loc. mart.}$$

(ii) If $H \in L^2_{loc}(M)$ and K is predictable then $K \in L^2_{loc}(H \bullet M)$ iff $HK \in L^2_{loc}(M)$ and then

$$H \bullet (K \bullet M) = (HK) \bullet M.$$

(iii) If T is a stopping time,

$$(\mathbb{1}_{[0,T]} H) \bullet M = (H \bullet M)^T = H \bullet M^T.$$

Finally, if $M \in \mathcal{M}^2_c$ and $H \in L^2(M)$ then the definition of $H \bullet M$ is the same as in the previous section.

Proof. (i) Assume $M_0 = 0$ and (setting $H=0$ where this fails)

$$\int_0^t H_s^2 \, d\langle M \rangle_s < \infty \quad \forall t \geq 0 \quad \forall \omega \in \Omega.$$

Set

$$S_n = \inf\left\{ t \geq 0 : \int_0^t (1 + H_s^2) \, d\langle M \rangle_s > n \right\}.$$

Note that S_n is a stopping time and $S_n \uparrow \infty$ as $n \to \infty$.

$$\langle M^{S_n}, M^{S_n} \rangle_t = \langle M, M \rangle_{t \wedge S_n} \leq n$$

$\Rightarrow$ $M^{S_n} \in \mathcal{M}_c^2$ and

$$\int_0^\infty H_s^2 \, d\langle M^{S_n} \rangle_s = \int_0^{S_n} H_s^2 \, d\langle M \rangle_s \leq n$$

$\Rightarrow$ $H \in L^2(M^{S_n})$ and $H \cdot M^{S_n}$ is already defined.

$(H \cdot M^{S_n}) = (H \cdot M^{S_m})^{S_n}$ for $m > n$ since stopping 'commutes' with $\cdot$

$\Rightarrow$ There is a unique process denoted $H \cdot M$ s.t.

$$(H \cdot M)^{S_n} = H \cdot M^{S_n} \quad \forall n$$

$H \cdot M$ is adapted, continuous, and is a local martingale since the $(H \cdot M)^{S_n}$ are martingales.

Claim: $\langle H\cdot M, N\rangle = H\cdot\langle M,N\rangle$

Assume that $N_0=0$ and set

$$S_n' = \inf\{t\geq 0 : |N_t|>n\}, \quad T_n = S_n\wedge S_n'.$$

$\Rightarrow N^{S_n'}\in \mathcal{M}_c^2$ and

$$\begin{aligned}\langle H\cdot M, N\rangle^{T_n} &= \langle (H\cdot M)^{S_n}, N^{S_n'}\rangle \\ &= \langle H\cdot M^{S_n}, N^{S_n'}\rangle \\ &= H\cdot\langle M^{S_n}, N^{S_n'}\rangle \\ &= H\cdot \langle M,N\rangle^{T_n} \\ &= (H\cdot\langle M,N\rangle)^{T_n}\end{aligned}$$

Since $T_n\uparrow\infty$ thus $\langle H\cdot M, N\rangle = H\cdot\langle M,N\rangle$.

Uniqueness follows as before.

Also, (ii) & (iii) follow as in the proof for $M\in\mathcal{M}_c^2$, $H\in L^2(M)$ since these only use (i).

If $M\in\mathcal{M}_c^2$ and $H\in L^2(M)$, then $H\cdot M\in\mathcal{M}_c^2$ by (i) which shows $\langle H\cdot M\rangle_\infty = (H^2\cdot\langle M\rangle)_\infty$ and thus $\|H\cdot M\|_{\mathcal{M}^2}^2 = \mathbb{E}\langle H\cdot M\rangle_\infty<\infty$.
The uniqueness statement in the equivalent of (i) from the previous statement thus shows consistency with previous definition.

3.4. Extension to semimartingales

Defn. A process H is locally bounded if

$$\forall t \geq 0: \sup_{s \leq t} |H_s| < \infty \quad \text{a.s.}$$

In particular, any continuous process is locally bounded.

Fact. If H is locally bounded and predictable and A is a finite variation process, then

$$\forall t \geq 0: \int_0^t |H_s| \, |dA_s| < \infty \quad \text{a.s.}$$

In particular, for such H, and M a continuous local martingale, one has $H \in L^2_{loc}(M)$.

Defn. Let $X = X_0 + M + A$ be a continuous semimartingale, and let H be a locally bounded process. Then the Itô integral $H \cdot X$ is the continuous semimartingale

$$H \cdot X = \underbrace{H \cdot M}_{\text{Itô integral}} + \underbrace{H \cdot A}_{\text{Lebesgue-Stieltjes integral}}$$

and we again write

$$(H \cdot X)_t = \int_0^t H_s \, dX_s .$$

Prop. (Stochastic DCT). Let X be a continuous semimartingale, and let H be a locally bounded predictable process, and let K be predictable and nonnegative. Let $t>0$ and assume that a.s.

(i) $H^n_s \xrightarrow{n\to\infty} H_s$ for all $s\in[0,t]$

(ii) $|H^n_s| \le K_s$ for all $s\in[0,t]$ and $n\in\mathbb{N}$

(iii) $\int_0^t K_s^2 \, d\langle M\rangle_s + \int_0^t K_s |dA_s| < \infty$ where $X = X_0 + M + A$.

(This condition is always satisfied if K is locally bounded.)

Then
$$\int_0^t H^n_s \, dX_s \xrightarrow{ucp} \int_0^t H_s \, dX_s .$$

Proof. Let $X = X_0 + A + M$. For the finite variation part A, the usual DCT implies
$$\int_0^t H^n_s \, dA_s \longrightarrow \int_0^t H_s \, dA_s .$$

Set
$$T_m = \inf\{t \ge 0 : \int_0^t K_s^2 \, d\langle M\rangle_s > m\}.$$

Then
$$\mathbb{E}\left(\left(\int_0^{T_m\wedge t} H^n_s \, dM_s - \int_0^{T_m\wedge t} H_s \, dM_s\right)^2\right) \underset{\text{Itô isometry}}{=} \mathbb{E}\left(\int_0^{T_m\wedge t} (H^n_s - H_s)^2 \, d\langle M\rangle_s\right)$$

$\to 0$ by DCT

Since $T_m \wedge t = t$ eventually, a.s., $P(T_m\wedge t = t) \to 0$ and the convergence follows for fixed t. For convergence ucp in t, use Doob's ineq.

Cor. Let X be a continuous semimartingale, and let H be a locally bounded adapted left-continuous process. Then for any subdivision $0=t_0^{(m)}<\cdots<t_{n_m}^{(m)}=t$ of $[0,t]$ with $\max_i |t_i^{(m)}-t_{i-1}^{(m)}|\to 0$,

$$\lim_{m\to\infty}\sum_{i=1}^{n_m} H_{t_{i-1}^{(m)}}\left(X_{t_i^{(m)}}-X_{t_{i-1}^{(m)}}\right) \overset{ucp}{=} \int_0^t H_s\,dX_s .$$

Proof. Exactly as in finite variation case using stochastic DCT.

Rk. Suppose H is continuous. Unlike the case that X is of finite variation, it is here essential that H is evaluated at the left end point of the interval $(t_{i-1}^{(m)},t_i^{(m)}]$.

Example.

$$\lim_{m\to\infty}\sum_{i=1}^{n_m} X_{t_{i-1}^{(m)}}\left(X_{t_i^{(m)}}-X_{t_{i-1}^{(m)}}\right) = \int_0^t X_s\,dX_s$$

but

$$\begin{aligned}\lim_{m\to\infty}\sum_{i=1}^{n_m} X_{t_i^{(m)}}\left(X_{t_i^{(m)}}-X_{t_{i-1}^{(m)}}\right) &= \lim_{m\to\infty}\sum_{i=1}^{n_m} X_{t_{i-1}^{(m)}}\left(X_{t_i^{(m)}}-X_{t_{i-1}^{(m)}}\right) \\ &\quad+\lim_{m\to\infty}\sum_{i=1}^{n_m}\left(X_{t_i^{(m)}}-X_{t_{i-1}^{(m)}}\right)^2 \\ &= \int_0^t X_s\,dX_s + \langle X,X\rangle_t\end{aligned}$$

For example, if $X=B$ is BM, then $\langle X,X\rangle_t=t$.

Rk. The choice of the left-end point gives the Itô integral. But we could have made a different choice from the start. When X and Y are continuous semimartingales, the Stratonovich integral is defined by

$$\int_0^t X_s \partial Y_s = \int_0^t X_s \, dY_s + \frac{1}{2}\langle X, Y\rangle_t .$$

It thus corresponds to the approximation

$$\sum_{i=1}^{n_m} \frac{1}{2}\left(X_{t_{i-1}^{(m)}} + X_{t_i^{(m)}}\right)\left(Y_{t_i^{(m)}} - Y_{t_{i-1}^{(m)}}\right).$$

Note that $\int_0^{\cdot} X_s \partial Y_s$ is generally not a martingale.

3.5. Itô formula

Thm (Integration by parts). Let X, Y be continuous semimartingales. Then a.s., for all t,

$$X_t Y_t - X_0 Y_0 = \int_0^t X_s \, dY_s + \int_0^t Y_s \, dX_s + \langle X, Y \rangle_t$$

Rk. $\langle X, Y \rangle_t$ is called the Itô correction. It is absent when X or Y is of finite variation. Also, in terms of the Stratonovich integral,

$$X_t Y_t - X_0 Y_0 = \int_0^t X_s \, \partial Y_s + \int_0^t Y_s \, \partial X_s .$$

Proof. Clearly,

$$X_t Y_t - X_s Y_s = X_s (Y_t - Y_s) + (X_t - X_s) Y_s + (X_t - X_s)(Y_t - Y_s).$$

Thus

$$\begin{aligned} X_{k2^{-m}} Y_{k2^{-m}} - X_0 Y_0 &= \sum_{i=1}^{k} \left(X_{i2^{-m}} Y_{i2^{-m}} - X_{(i-1)2^{-m}} Y_{(i-1)2^{-m}} \right) \\ &= \sum_{i=1}^{k} \Big(X_{(i-1)2^{-m}} \left(Y_{i2^{-m}} - Y_{(i-1)2^{-m}} \right) \\ &\qquad + \left(X_{i2^{-m}} - X_{(i-1)2^{-m}} \right) Y_{(i-1)2^{-m}} \\ &\qquad + \left(X_{i2^{-m}} - X_{(i-1)2^{-m}} \right) \left(Y_{i2^{-m}} - Y_{(i-1)2^{-m}} \right) \Big) \end{aligned}$$

Thus for $t \in 2^{-n}\mathbb{N}$, by taking $m \to \infty$,

$$X_t Y_t - X_0 Y_0 = (X \cdot Y)_t + (Y \cdot X)_t + \langle X, Y \rangle_t .$$

For $t \in \mathbb{R}$ use continuity of both sides.

Thm (Itô's formula). Let $X^1, \dots, X^p$ be continuous semimartingales, and let $f: \mathbb{R}^p \to \mathbb{R}$ be C^2. Then, writing $X = (X^1, \dots, X^p)$, a.s.,

$$f(X_t) = f(X_0) + \underbrace{\sum_{i=1}^{p} \int_0^t \frac{\partial f}{\partial x^i}(X_s)\, dX_s^i}_{\text{cont. loc. mart. + f.v. process}} + \underbrace{\frac{1}{2} \sum_{i,j=1}^{p} \int_0^t \frac{\partial^2 f}{\partial x^i \partial x^j}(X_s)\, d\langle X^i, X^j \rangle_s}_{\text{f.v. process}} \quad (*)$$

Proof. For f constant, $(*)$ is obvious.

Claim: Assume $(*)$ holds for some f. Then it also holds for g defined by $g(x) = x^k f(x)$.
x^k: k-th component.

Apply IBP with $X = X^k$ and $Y = f(X)$:

$$g(X_t) - g(X_0) = \int_0^t X_s^k \, df(X_s) + \int_0^t f(X_s)\, dX_s^k + \langle X^k, f(X) \rangle_t .$$

By $(*)$ for f and $H \cdot (K \cdot X) = (HK) \cdot X$,

$$\int_0^t X_s^k \, df(X_s) = (X^k \cdot f(X))_t = \sum_{i=1}^{p} \int_0^t X_s^k \frac{\partial f}{\partial x^i}(X_s)\, dX_s^i + \frac{1}{2} \sum_{i,j=1}^{p} \int_0^t X_s^k \frac{\partial^2 f}{\partial x^i \partial x^j}(X_s)\, d\langle X^i, X^j \rangle_s$$

By $(*)$ for f and $\langle X, H \cdot Y \rangle = H \cdot \langle X, Y \rangle$,

$$\langle X^k, f(X) \rangle_t = \sum_{i=1}^{p} \int_0^t X_s^k \frac{\partial f}{\partial x^i}(X_s)\, d\langle X^k, X^i \rangle_s$$

$$\Rightarrow g(X_t) = g(X_0) + \sum_{i=1}^{p} \int_0^t \frac{\partial g}{\partial x^i}(X_s)\, dX_s^i + \frac{1}{2}\sum_{i,j=1}^{p} \int_0^t \frac{\partial^2 g}{\partial x^i \partial x^j}(X_s)\, d\langle X^i, X^j\rangle_s .$$

By induction, $(*)$ holds for all polynomials.

Write $X^i = X_0^i + M^i + A^i$.

Claim: $(*)$ holds for $f \in C^2$ if $|X_t^i(\omega)| \le n$, $\int_0^t |dA_s^i| \le n$ $\forall (t,\omega)$.

By the Weierstraß approximation theorem, there are polynomials p_k s.t.

$$\sup_{|x| \le k} \left(|f(x) - p_k(x)| + |\nabla f(x) - \nabla p_k(x)| + |\nabla^2 f(x) - \nabla^2 p_k(x)| \right) \le \frac{1}{k}.$$

Taking limits, in probability,

$$f(X_t) - f(X_0) = \lim_{k\to\infty} \left(p_k(X_t) - p_k(X_0) \right)$$

$$\int_0^t \frac{\partial f}{\partial x^i}(X_s)\, dX_s^i = \lim_{k\to\infty} \int_0^t \frac{\partial p_k}{\partial x^i}\, dX^i \quad \text{by stochastic DCT}$$

$$\int_0^t \frac{\partial^2 f}{\partial x^i \partial x^j}(X_s)\, d\langle X^i, X^j\rangle_s = \lim_{k\to\infty} \int_0^t \frac{\partial^2 p_k}{\partial x^i \partial x^j}\, d\langle X^i, X^j\rangle_s \quad \text{by DCT.}$$

Claim: $(*)$ holds without restriction on X.

Let $T_n = \inf\{t \ge 0 : \max_i |X_t^i| \ge n, \max_i \int_0^t |dA_s| \ge n\}$. Then (by above)

$$f(X_t^{T_n}) = f(X_0^{T_n}) + \sum_{i=1}^{p} \int_0^t \frac{\partial f}{\partial x^i}(X_s^{T_n})\, d(X^i)_s^{T_n} + \frac{1}{2}\sum_{i,j} \int_0^t \frac{\partial^2 f}{\partial x^i \partial x^j}(X_s^{T_n})\, d\langle (X^i)^{T_n}, (X^j)^{T_n}\rangle_s$$

$$= f(X_0) + \sum_i \int_0^{t\wedge T_n} \frac{\partial f}{\partial x^i}(X_s)\, dX_s^i + \frac{1}{2}\sum_{i,j} \int_0^{t\wedge T_n} \frac{\partial^2 f}{\partial x^i \partial x^j}(X_s)\, d\langle X^i, X^j\rangle_s$$

Take $n \to \infty$.

Rk. In terms of the Stratonovich integral,

$$f(X_t) = f(X_0) + \sum_{i=1}^{p} \int_0^t \frac{\partial f}{\partial x^i}(X_s)\, \partial X_s^i$$

Rk. Formally, one can write Itô's formula as

$$df(X_t) = \sum_i \frac{\partial f}{\partial x^i}\, dX^i + \frac{1}{2}\sum_{i,j} \frac{\partial^2 f}{\partial x^i \partial x^j}\, d\langle X^i, X^j\rangle .$$

Thus one formally Taylor expands f using "$dX^i dX^j = d\langle X^i, X^j\rangle$" and that higher powers vanish.

If $X = B$ is BM, thus "$(dB)^2 = dt$" and

$$df(B_t) = f'(B_t)\, dB_t + \frac{1}{2} f''(B_t)\, dt .$$

The following computational rules hold: with

$$Z_t - Z_0 = \int_0^t H_s\, dX_s \iff dZ_t = H_t\, dX_t$$

$$Z_t - Z_0 = \langle X, Y\rangle_t = \int_0^t d\langle X, Y\rangle_s \iff dZ_t = dX_t\, dY_t$$

one has

- $H_t (K_t\, dX_t) = (H_t K_t)\, dX_t$ (i.e. $H\cdot(K\cdot X) = (HK)\cdot X$) (associativity)
- $H_t (dX_t\, dY_t) = (H_t\, dX_t)\, dY_t$ (i.e. $H\cdot\langle X, Y\rangle = \langle H\cdot X, Y\rangle$) (Kunita–Watanabe)
- $d(X_t Y_t) = X_t\, dY_t + Y_t\, dX_t + dX_t\, dY_t$ (IBP)
- $df(X_t) = \sum \frac{\partial f}{\partial x^i}(X_t)\, dX_t^i + \frac{1}{2}\sum \frac{\partial^2 f}{\partial x^i \partial x^j}(X_t)\, dX_t^i\, dX_t^j$ (Itô formula)

Cor. Let $\Omega \subset \mathbb{R}^d$ be open and $f: \Omega \to \mathbb{R}$ be in $C^2(\overline{\Omega})$. Let X be a semimartingale s.t. $X_0 \in \Omega$ a.s. and let $T = \inf\{t \geq 0 : X_t \notin \Omega\}$. Then a.s. for all $t < T$,

$$f(X_t) = \sum_{i=1}^{d} \int_0^t \frac{\partial f}{\partial x^i}(X_s)\, dX_s^i + \frac{1}{2}\sum_{i,j=1}^{d} \int_0^t \frac{\partial^2 f}{\partial x^i \partial x^j}(X_s)\, d\langle X^i, X^j \rangle_s .$$

Proof. Exercise.

4. Applications to Brownian motion and martingales

4.1. Levy's characterisation of Brownian motion

<u>Thm.</u> Let $X = (X^1, \dots, X^d)$ be continuous local martingales. Suppose that $X_0 = 0$ and that $\langle X^i, X^j \rangle_t = \delta_{ij} t$ for all $t \geq 0$. Then X is a standard d-dimensional Brownian motion.

<u>Proof</u>. It suffices to show that for all $0 \leq s < t$, $X_t - X_s$ is independent of $\mathcal{F}_s$ and that $X_t - X_s \sim \mathcal{N}(0, (t-s)\, id_{d\times d})$. Both properties follow from the following claim:

(*) $\mathbb{E}\left(e^{i\theta \cdot (X_t - X_s)} \,|\, \mathcal{F}_s\right) = e^{-\frac{1}{2}|\theta|^2 (t-s)}$ for all $\theta \in \mathbb{R}^d$, $s < t$.

(Indeed, (*) implies $\mathbb{E}(e^{i\theta \cdot (X_t - X_s)}) = e^{-\frac{1}{2}|\theta|^2(t-s)}$ so that $X_t - X_s \sim \mathcal{N}(0, (t-s)\, id_{d\times d})$. To show independence, consider $A \in \mathcal{F}_s$ with $\mathbb{P}(A) > 0$. Then $X_t - X_s \sim \mathcal{N}(0, (t-s) id_{d\times d})$ under the probability measure $\mathbb{P}(\cdot \,|A) = \mathbb{P}(\cdot \cap A)/\mathbb{P}(A)$. Hence

$$\mathbb{E}(1_A\, f(X_t - X_s)) = \mathbb{P}(A)\, \mathbb{E}(f(X_t - X_s)), \quad \forall f$$

i.e. A is independent of $X_t - X_s$. The same is trivial if $\mathbb{P}(A) = 0$. Thus $X_t - X_s$ is independent of $\mathcal{F}_s$.)

To show (*), fix $\theta \in \mathbb{R}^d$ and set $Y_t = \theta \cdot X_t = \sum_{j=1}^d \theta^j X_t^j$.

$$\Rightarrow \langle Y \rangle_t = \langle Y, Y \rangle_t = \sum_{j,k=1}^d \theta^j \theta^k \langle X^j, X^k \rangle_t = |\theta|^2 t \text{ by assumption.}$$

Let $Z_t = e^{iY_t + \frac{1}{2}\langle Y\rangle_t} = e^{i\theta\cdot X_t + \frac{1}{2}|\theta|^2 t}$. By Itô's formula applied with $X = iY + \frac{1}{2}\langle Y\rangle$ and $f(x) = e^x$,

$$dZ_t = df(X_t) = Z_t\left(i\,dY_t + \frac{1}{2}\langle Y\rangle_t - \frac{1}{2}\langle Y\rangle_t\right) = i Z_t dY_t,$$

i.e.

$$Z_t - Z_0 = i\int_0^t Z_s\, dY_s.$$

$\Rightarrow Z$ is a continuous local martingale.

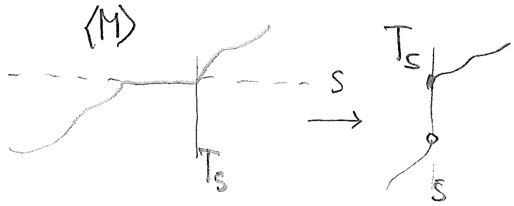
Since Z is bounded on every bounded interval, Z is in fact a martingale.

$$\Rightarrow \mathbb{E}(Z_t \mid \mathcal{F}_s) = Z_s$$

$$\Rightarrow \mathbb{E}\left(e^{i\theta\cdot(X_t - X_s)}\right) = e^{-\frac{1}{2}|\theta|^2(t-s)}.$$

4.2. Dubins–Schwarz Theorem

<u>Thm</u>. Let M be a continuous local martingale with $M_0 = 0$ and $\langle M \rangle_\infty = \infty$ a.s. Let



$$T_s = \inf\{t \geq 0 : \langle M \rangle_t > s\}$$

be the right-continuous inverse of $t \mapsto \langle M \rangle_t$. Let

$$B_s = M_{T_s}, \quad \mathcal{G}_s = \mathcal{F}_{T_s}.$$

Then T_s is an $(\mathcal{F}_t)$-stopping time, $\langle M \rangle_{T_s} = s$ for all $s \geq 0$, B is a $(\mathcal{G}_s)$-Brownian motion, and

$$M_t = B_{\langle M \rangle_t},$$

i.e., M is a (random) time change of B.

<u>Lemma</u>. A.s. for all $u < v$,

M is constant on $[u,v]$ $\Leftrightarrow$ $\langle M \rangle$ is constant on $[u,v]$.

<u>Proof</u>. By continuity, it suffices to prove that for any fixed $u < v$ one has a.s.

$$\{M_t = M_u \ \forall t \in [u,v]\} = \{\langle M \rangle_v = \langle M \rangle_u\}.$$

Let $N_t = M_t - M_{t \wedge u} = \int_{t \wedge u}^{t} dM_s$. Then $\langle N \rangle_t = \int_{t \wedge u}^{t} d\langle N \rangle_s = \langle N \rangle_t - \langle N \rangle_{t \wedge u}$.

Let $T_\varepsilon = \inf\{t \geq 0 : \langle N \rangle_t > \varepsilon\}$.

$\Rightarrow N^{T_\varepsilon} \in \mathcal{M}_c^2$ since $\langle N^{T_\varepsilon}\rangle \leq \varepsilon$.

$\mathbb{E}(N^{T_\varepsilon})^2 = \mathbb{E}\langle N^{T_\varepsilon}\rangle \leq \varepsilon$.

$$\Rightarrow \mathbb{E}\left(1_{\{\langle M\rangle_v = \langle M\rangle_u\}} N_t^2\right) = \mathbb{E}\left(1_{\langle N\rangle_v = 0} N_t^2\right) \qquad (t \in [u,v])$$

$$= \mathbb{E}\left(1_{\langle N\rangle_v = 0} N_{t\wedge T_\varepsilon}^2\right) \leq \varepsilon \quad \forall \varepsilon > 0.$$

$\Rightarrow N_t = 0$ a.s. on $\{\langle M\rangle_v = \langle M\rangle_u\}$ for $t \in [u,v]$

$\Rightarrow \langle M\rangle_v = \langle M\rangle_u$ implies M is constant on $[u,v]$, a.s.

Other direction: exercise (use approximation for example).

Proof of Dubins-Schwarz Theorem.

T_s is a stopping time and $T_s < \infty$ for all s, a.s., since $\langle M\rangle$ is continuous increasing adapted a.s.

Redefine $T_s = 0$ if $\langle M\rangle_\infty < \infty$.

Claim: $(\mathcal{G}_s)$ is a filtration obeying the usual conditions and $\mathcal{G}_\infty = \mathcal{F}_\infty$.

$$A \in \mathcal{G}_s \Rightarrow A \cap \{T_t \leq u\} = A \cap \{T_s \leq u\} \cap \{T_t \leq u\} \in \mathcal{F}_u$$

$$\Rightarrow A \in \mathcal{F}_{T_t} = \mathcal{G}_t \quad \text{for any } t \geq s$$

$\Rightarrow (\mathcal{G}_s)$ is a filtration

Right-continuity follows from that of $(\mathcal{F}_t)$ and $t \mapsto T_t$.

Claim: B is adapted to $(\mathcal{G}_s)$.

If X is càdlàg and T a stopping time then $X_T 1_{\{T<\infty\}} \in \mathcal{F}_T$
($\rightarrow$ A.P.)

Apply this with $X = M$, $T = T_s$, $\mathcal{F}_T = \mathcal{G}_s$. Thus $B_s \in \mathcal{G}_s$.

Claim: B is continuous.

T_s is càdlàg in $s \Rightarrow B_s = M_{T_s}$ is càdlàg and thus right-cont.

B is left-continuous $\Leftrightarrow B_s = B_{s-} \Leftrightarrow M_{T_s} = M_{T_{s-}}$ where
$$T_{s-} = \inf\{t \geq 0 : \langle M \rangle_t = s\}.$$

Hence if $T_s = T_{s-}$ then B is left-continuous.

If $T_s > T_{s-}$ then $\langle M \rangle$ is constant on $[T_{s-}, T_s]$.

By the previous lemma, thus M is constant on $[T_{s-}, T_s]$, a.s.

Hence $M_{T_s} = M_{T_{s-}}$ holds as well and B is left-continuous.

Claim: B is a continuous martingale with respect to $(\mathcal{G}_s)$ and $\langle B \rangle_s = s$ for all $s \geq 0$.

Let $0 \leq r < s$. Then $\langle M^{T_s} \rangle_\infty = \langle M \rangle_{T_s} = s$
$$\Rightarrow M^{T_s} \in \mathcal{M}^2_c$$
$$\Rightarrow (M^2 - \langle M \rangle)^{T_s} \text{ is a UI martingale}$$

OST implies

$$\mathbb{E}(B_s \mid \mathcal{G}_r) = \mathbb{E}(M_\infty^{T_s} \mid \mathcal{F}_{T_r}) = M_{T_r} = B_r$$

$$\mathbb{E}(B_s^2 - s \mid \mathcal{G}_r) = \mathbb{E}((M^2 - \langle M \rangle)^{T_s} \mid \mathcal{F}_{T_r}) = M_{T_r}^2 - \langle M \rangle_{T_r} = B_r^2 - r$$

$\Rightarrow$ B is a continuous martingale with $\langle B \rangle_s = s$.

By Lévy's characterisation, it follows that B is a BM.

4.3. Girsanov's Theorem

<u>Example</u>. Let $X \sim \mathcal{N}(0,C)$ be an n-dimensional Gaussian vector with positive definite covariance matrix $C=(C_{ij})_{i,j=1}^n$ and mean 0:

$$\mathbb{E}(f(X)) = \left(\det \frac{M}{2\pi}\right)^{1/2} \int_{\mathbb{R}^n} f(x)\, e^{-\frac{1}{2}(x,Mx)}\, dx, \quad M = C^{-1}.$$

Let $a \in \mathbb{R}^n$. Then

$$\mathbb{E}(f(X+a)) = \left(\det \frac{M}{2\pi}\right)^{1/2} \int_{\mathbb{R}^n} f(x)\, \underbrace{e^{-\frac{1}{2}(x-a,M(x-a))}}_{e^{-\frac{1}{2}(x,Mx)}\,\underbrace{e^{-\frac{1}{2}(a,Ma)+(x,Ma)}}_{Z}}\, dx$$

$$= \mathbb{E}(Z f(X)).$$

Thus if $\mathbb{P}$ denotes the distribution of X then the measure $\mathbb{Q}$ with

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = Z$$

is that of a $\mathcal{N}(a,C)$ Gaussian vector.

<u>Example</u>. Let B be a standard BM with $B_0=0$. Fix finitely many times $0=t_0<t_1<\dots<t_n$. Then $(B_{t_i})_{i=1}^n$

is a centred Gaussian vector with

$$\mathbb{E}(f(B_{t_i})) = \text{const.} \int_{\mathbb{R}^n} f(x)\, e^{-\frac{1}{2}\sum_{i=1}^n \frac{(x_i - x_{i-1})^2}{t_i - t_{i-1}}}\, dx_1 \cdots dx_n .$$

Let $h: \mathbb{R}_+ \to \mathbb{R}$ be a deterministic function. Then

$$\mathbb{E}(f((B+h)_{t_i})) = \mathbb{E}(Z\, f((B_{t_i}))),$$

with

$$Z = \exp\left(-\frac{1}{2}\sum_{i=1}^n \frac{(h_{t_i} - h_{t_{i-1}})^2}{t_i - t_{i-1}} + \sum_{i=1}^n \frac{(h_{t_i} - h_{t_{i-1}})(B_{t_i} - B_{t_{i-1}})}{t_i - t_{i-1}} \right).$$

Girsanov's theorem generalises this example to general functions of continuous local martingales.

<u>Defn</u>. Let M be a continuous local martingale. Then the stochastic exponential of M is

$$\mathcal{E}(M)_t = e^{M_t - \frac{1}{2}\langle M \rangle_t} .$$

<u>Fact</u>. Let M be a cont. local martingale with $M_0 = 0$. Then $Z = \mathcal{E}(M)$ is a continuous local martingale and satisfies

$$dZ_t = Z_t\, dM_t, \quad \text{i.e.,} \quad Z_t = 1 + \int_0^t Z_s\, dM_s .$$

Proof. By Itô's formula applied to $X = M - \frac{1}{2}\langle M\rangle$ and $f(x) = e^x$,

$$dZ_t = df(X_t) = Z_t\left(dM_t - \frac{1}{2}d\langle M\rangle_t + \frac{1}{2}d\langle M\rangle_t\right) = Z\,dM_t .$$

Since M is a continuous local martingale, so is $Z\cdot M$. Hence Z is a continuous local martingale.

Thm (Girsanov). Let L be a continuous local martingale with $L_0 = 0$. Suppose that $\mathcal{E}(L)$ is a uniformly integrable martingale. Define a probability measure $\mathbb{Q}$ by

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \mathcal{E}(L)_\infty .$$

Then if M is a continuous local martingale w.r.t. $\mathbb{P}$, $\tilde{M} = M - \langle M, L\rangle$ is a continuous local martingale w.r.t. $\mathbb{Q}$.

Rk. The quadratic variation does not change, i.e., $\langle M\rangle = \langle \tilde{M}\rangle$. This follows for example from

$$\langle M\rangle_t = \lim_{n\to\infty} \sum_{i=1}^{\lfloor 2^n t\rfloor} \left(M_{2^{-n}i} - M_{2^{-n}(i-1)}\right)^2 \quad \text{a.s. along a subseq.}$$

Proof. Let

$$T_n = \inf\{t \geq 0 : |\hat{M}_t| \geq n\}.$$

Then T_n is a stopping time and $\mathbb{P}(T_n \uparrow \infty) = 1$ by continuity of $\hat{M}$. Since $\mathbb{Q}$ is absolutely continuous w.r.t. $\mathbb{P}$, also $\mathbb{Q}(T_n \uparrow \infty) = 1$.

Thus it suffices to show that $\hat{M}^{T_n}$ is a continuous martingale w.r.t. $\mathbb{Q}$ for every n.

Let $Y_t = M_t^{T_n} - \langle M^{T_n}, L\rangle_t$ and $Z_t = \mathcal{E}(L)_t$.

Claim: ZY is a continuous local martingale w.r.t. $\mathbb{P}$.

$$d(ZY) = Y_t\, dZ_t + Z_t\, dY_t + d\langle Z, Y\rangle_t \qquad (\text{Itô})$$

$$= \left(M_t^{T_n} - \langle M^{T_n}, L\rangle_t\right) Z_t\, dL_t + Z_t\left(dM_t^{T_n} - \cancel{d\langle M^{T_n}, L\rangle_t}\right)$$
$$+ Z_t\, \cancel{d\langle L, M^{T_n}\rangle_t}$$

where we used that

$$d\langle Z, Y\rangle = d\langle Z, M^{T_n}\rangle \underset{\uparrow}{=} Z\, d\langle L, M^{T_n}\rangle$$

$dZ = Z\,dL$ and $\langle Z \cdot L, M^{T_n}\rangle = Z \cdot \langle L, M^{T_n}\rangle$

Thus $d(ZY)$ is a sum of stochastic differentials w.r.t. local martingales.

Hence ZY is a local martingale.

Claim: ZY is a UI martingale w.r.t. $\mathbb{P}$.

This follows from the fact that $Z = \mathcal{E}(L)$ is by assumption a UI martingale and that $|Y| \leq n$ is bounded.

Indeed, a local martingale M is a martingale iff

$\forall t: \mathcal{X}_t = \{ M_T : T \text{ is a stopping time with } T \leq t \}$ is UI.

(Exercise, see p. 22)

Note that uniform integrability is stable under multiplication by bounded random variables.

Claim: Y is a martingale w.r.t. $\mathbb{Q}$

$$\mathbb{E}^{\mathbb{Q}}(Y_t - Y_s \mid \mathcal{F}_s) = \mathbb{E}^{\mathbb{P}}(Z_\infty Y_t - Z_\infty Y_s \mid \mathcal{F}_s)$$
$$= \mathbb{E}^{\mathbb{P}}(Z_t Y_t - Z_s Y_s \mid \mathcal{F}_s) = 0.$$

Since $Y = (M - \langle M, L \rangle)^{T_n}$ is a martingale and $T_n \uparrow \infty$ a.s., thus $M - \langle M, L \rangle$ is a local martingale.

<u>Prop</u>. Suppose that $\langle L \rangle$ is bounded : $\langle L \rangle_\infty \le C$. Then $\mathcal{E}(L)$ is a UI martingale.

<u>Proof</u>. It suffices to show that $\sup_t L_t$ has Gaussian tail;

$$\mathbb{P}\Big(\sup_{t \ge 0} L_t \ge a\Big) = \mathbb{P}\Big(\sup_{t\ge 0} L_t \ge 0, \langle L \rangle_\infty \le C\Big) \le e^{-\frac{a^2}{2C}} \qquad (*)$$

Indeed, then

$$\mathbb{E}\Big(\sup_t \mathcal{E}(L)_t\Big) \overset{\langle L\rangle \ge 0}{\le} \mathbb{E}\Big(\exp\big(\sup L_t\big)\Big)$$
$$= \int_0^\infty \mathbb{P}\big(\exp(\sup L_t) \ge \lambda\big)\, d\lambda$$
$$= \int_0^\infty \mathbb{P}\big(\sup L_t \ge \log \lambda\big)\, d\lambda$$
$$\le 1 + \int_1^\infty e^{-\frac{(\log \lambda)^2}{2b}}\, d\lambda < \infty$$

so $\mathcal{E}(L)_t$ is bounded by the random variable $\sup \mathcal{E}(L)_t \in L^1$ and thus $\mathcal{E}(L)$ is a uniformly integrable martingale.

To show $(*)$, let $T = \inf\{t \ge 0 : L_t \ge a\}$. Then

$$Z_t = e^{\theta L^T_t - \frac{1}{2}\theta^2 \langle L^T \rangle}$$

is a bounded martingale. Hence $EZ_\infty = Z_0 = 1$. Thus

$$\mathbb{P}\Big(\sup_t L_t \ge a, \langle L \rangle_\infty \le C\Big) \le \mathbb{P}\Big(L^T_\infty \ge a, \langle L^T \rangle_\infty \le C\Big)$$
$$= \mathbb{P}\Big(Z_\infty \ge e^{\theta a - \frac{1}{2}\theta^2 C}\Big) \le e^{-\theta a + \frac{1}{2}\theta^2 C}$$

Take inf over θ.

A more general criterion is Novikov's.

<u>Thm</u> (Novikov). Let M be a continuous local martingale with $M_0=0$. Then

$$\mathbb{E}\left(e^{\frac{1}{2}\langle M\rangle_\infty}\right) < \infty$$

implies that $\mathcal{E}(M)$ is a UI martingale.

We will not prove this for now.

<u>Cor.</u> (of Girsanov's Theorem). Let B be a standard Brownian under $\mathbb{P}$, and let L be a continuous local martingale with $L_0=0$ s.t. $\mathcal{E}(L)$ is UI. Then $\tilde{B}=B-\langle B,L\rangle$ is a standard Brownian motion under $\mathbb{Q}$ where $d\mathbb{Q}/d\mathbb{P}=\mathcal{E}(L)_\infty$.

<u>Proof</u>. By Girsanov's Theorem, $\tilde{B}$ is a continuous local martingale. Moreover,

$$\langle\tilde{B}\rangle_t = \langle B\rangle_t = t.$$

By Lévy's characterisation, $\tilde{B}$ is thus a standard BM.

Example. Consider the SDE

$$dX_t = b(X_t)\,dt + dB_t, \quad t \le T.$$

We can construct a solution as follows: Let X be a standard BM under $\mathbb{P}$. Set

$$L_t = \int_0^{t\wedge T} b(X_s)\,dX_s$$

Assume that $\mathcal{E}(L)$ is a UI martingale. Then

$$X_t - \langle X, L\rangle_t = X_t - \int_0^{t\wedge T} b(X_s)\,d\langle X\rangle_s = X_t - \int_0^{t\wedge T} b(X_s)\,ds$$

is a standard Brownian motion under $\mathbb{Q}$ given by $d\mathbb{Q}/d\mathbb{P} = \mathcal{E}(L)_\infty$. Thus if we call this BM B,

$$X_t - \int_0^{t\wedge T} b(X_s)\,ds = B_t$$

When is $\mathcal{E}(L)$ a UI martingale?

$$\langle L\rangle_\infty = \int_0^T b(X_s)^2\,ds$$

So it is for example sufficient if b is bounded.

4.4. The Cameron-Martin formula

<u>Defn</u>. The <u>Wiener space</u> $(W, \mathcal{W}, P)$ is given by $W = C(\mathbb{R}_+, \mathbb{R})$, $\mathcal{W} = \sigma(X_t : t \geq 0)$ where $X_t : W \to \mathbb{R}$ is given by $X_t(w) = w(t)$ and P is the unique probability measure on $(W, \mathcal{W})$ s.t. X_t is a standard Brownian motion. X is also called the canonical version of Brownian motion.

<u>Defn</u>. The <u>Cameron-Martin space</u> is

$$\mathcal{H} = \{ h \in W : h(t) = \int_0^t g(s)\, ds \text{ for some } g \in L^2(\mathbb{R}_+) \}.$$

For $h \in \mathcal{H}$, the function $\dot{h} = g$ is the weak derivative of h.

<u>Exercise</u>. $\mathcal{H}$ is a Hilbert space with inner product

$$(h, f)_{\mathcal{H}} = \int_0^\infty \dot{h}(s) \dot{f}(s)\, ds.$$

The dual space of $\mathcal{H}$ can be identified with

$$\mathcal{H}^* = \{ \mu \in M(\mathbb{R}_+) : \int_0^\infty (s \wedge t)\, \mu(ds)\, \mu(dt) = (\mu, \mu)_{\mathcal{H}^*} < \infty,\ \mu(\{0\}) = 0 \},$$

in the sense that for any bounded linear $\ell : \mathcal{H} \to \mathbb{R}$, there is $\mu \in \mathcal{H}^*$ s.t. $\ell(h) = \int_0^\infty h(t)\, \mu(dt)$ and vice versa.

We would like to think of Brownian motion as the standard Gaussian measure on $\mathcal{H}$. This measure does not exist, but the following theorem shows that this intuition is almost true.

<u>Thm</u> (Cameron–Martin). Let $h \in \mathcal{H}$ and define P^h by $P^h(A) = P(\{w \in W : w+h \in A\})$ for $A \in \mathcal{W}$. The measure P^h is absolutely continuous w.r.t. to Wiener measure P and

$$\frac{dP^h}{dP} = \exp\left(\int_0^\infty \dot{h}(s)\, dX_s - \frac{1}{2}\int_0^\infty \dot{h}(s)^2\, ds\right)$$

Wiener integral (= Itô integral)

<u>Proof</u>. Apply Girsanov with

$$L_t = \int_0^t \dot{h}(s)\, dX_s .$$

Since $\langle L \rangle_\infty = \int_0^\infty \dot{h}(s)^2\, ds = \|h\|^2_{\mathcal{H}} < \infty$, $\mathcal{E}(L)$ is a <u>UI</u> martingale. The rest is as in the last example in the previous section.

5. Stochastic differential equations

5.1. Notions of solutions

In Section 1, we considered the equation "$\dot{x}(t) = F(x(t)) + \eta(t)$"
white noise

Recall that the integral of white noise does not exist in the classical sense, but that we interpret it as Brownian motion. Thus it is reasonable to interpret the above equation as

$$X_t - X_0 = \int_0^t F(X_s)\, ds + B_t$$

$\Leftrightarrow$ $dX_t = F(X_t)\,dt + dB_t$ (where $\Leftrightarrow$ holds by notation)

Defn. Let $d, m \in \mathbb{N}$, $b: \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$, $\sigma: \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^{d \times m}$ be locally bounded Borel functions. The SDE $E(\sigma, b)$ is

$$dX_t = b(t, X_t)\, dt + \sigma(t, X_t)\, dB_t. \qquad (E(\sigma, b))$$

The SDE $E(\sigma, b)$ together with the initial condition $X_0 = x \in \mathbb{R}^d$ is denoted $E_x(\sigma, b)$.

Defn. A (weak) solution to the SDE $E(\sigma, b)$ consists of

- a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ obeying the usual conditions.

- an m-dimensional $(\mathcal{F}_t)$-Brownian motion
- an $(\mathcal{F}_t)$-adapted continuous process X with values in $\mathbb{R}^d$ such that

$$X_t = X_0 + \int_0^t \sigma(s, X_s)\, dB_s + \int_0^t b(s, X_s)\, ds.$$

Defn. For a strong solution to $E(\sigma, b)$, we specify the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a Brownian motion B, and choose $(\mathcal{F}_t)$, to be the completed filtration of B. A strong solution is then an $(\mathcal{F}_t)$-adapted continuous process X as in the definition of (weak) solution.

This means one can think of a strong solution as a function of the Brownian motion.

Defn. For the SDE we say that there is

- weak uniqueness (or uniqueness in law) if all solutions to $E_x(\sigma, b)$ have the same law.
- pathwise uniqueness if, for $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ and B fixed, all solutions are indistinguishable.

<u>Thm</u> (Yamada–Watanabe). Assume that $E_x(\sigma, b)$ has a weak solution and that pathwise uniqueness holds. Then:

- uniqueness in law holds
- for any $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ and B, there is a unique strong solution to $E_x(\sigma, b)$.

We will not need this (and therefore not prove it).

<u>Example</u> (Tanaka). The SDE

$$dX_t = \operatorname{sign}(X_t)\, dB_t, \quad X_0 = x, \quad \text{where } \operatorname{sign}(x) = \begin{cases} +1 & \text{if } x > 0 \\ -1 & \text{if } x \leq 0 \end{cases}$$

has a weak solution with weak uniqueness but not pathwise uniqueness.

<u>Proof</u>. Let X be a one-dimensional Brownian motion with $X_0 = x$. Set

$$B_t = \int_0^t \operatorname{sign}(X_s)\, dX_s .$$

$$\Rightarrow \; x + \int_0^t \operatorname{sign}(X_s)\, dB_s = x + \int_0^t \underbrace{\operatorname{sign}(X_s)^2}_{=1} dX_s = x + X_t - X_0 = X_t,$$

i.e.,

$$dX_t = \operatorname{sign}(X_t)\, dB_t, \quad X_0 = x.$$

Moreover, B is a Brownian motion since it is a continuous local martingale with $\langle B \rangle = \int_0^t d\langle X \rangle_s = t$.

If fact, any solution is a Brownian motion by the same argument. Thus there is weak uniqueness.

To show that pathwise uniqueness fails, we will show that if X is a solution with $X_0=0$ then $-X$ is also a solution with the same Brownian motion. Indeed,

$$-X_t = -\int_0^t \text{sign}(X_s)\, dB_s = \int_0^t \text{sign}(-X_s)\, dB_s + \underbrace{2\int_0^t 1_{X_s=0}\, dB_s}_{N_t}$$

where N_t is a continuous local martigale with

$$\langle N \rangle_t = 4\int_0^t 1_{X_s=0}\, ds = 0$$

since the zero set of Brownian motion has Lebesgue measure 0.

$\Rightarrow N=0$

$\Rightarrow -X$ also solves $(*)$.

Rk. X is not a strong solution.

<u>Thm.</u> Suppose that b and σ are locally Lipschitz, i.e., there exist $K_n > 0$ such that for all $|x|, |y| \le n$, $t \ge 0$,

$$|b(t,x) - b(t,y)| \le K_n |x-y|$$
$$|\sigma(t,x) - \sigma(t,y)| \le K_n |x-y|.$$

Then pathwise uniqueness holds for $E(\sigma, b)$

<u>Proof.</u> Let X and X' be two solutions to $E(\sigma, b)$ defined on the same probability space such that $X_0 = X_0'$.

Let $T_n = \inf\{t \ge 0 : |X_t| > n \text{ or } |X_t'| > n\}$,

and $f_n(t) = \mathbb{E}(|X_{t\wedge T_n} - X'_{t\wedge T_n}|^2)$.

By continuity of X and X', it suffices to show that for all n and t, one has $f_n(t) = 0$. Note that f_n is bounded.

Claim: $f_n(t) = 0$ for all n, t.

By Itô's formula,

$$|X_{t\wedge T_n} - X'_{t\wedge T_n}|^2 = \int_0^{T_n\wedge t} 2(X_s - X_s') \cdot (b(X_s) - b(X_s'))\, ds$$
$$+ \int_0^{T_n\wedge t} 2(X_s - X_s') \cdot (\sigma(X_s) - \sigma(X_s'))\, dB_s \ (\to \mathbb{E}[\cdot] = 0)$$
$$+ \int_0^{T_n\wedge t} |\sigma(X_s) - \sigma(X_s')|^2\, ds.$$

Since $(X_s - X'_s)(\sigma(X_s) - \sigma(X'_s))$ is bounded on $s < T_n$,

$$\int_0^{t \wedge T_n} 2(X_s - X'_s)(\sigma(X_s) - \sigma(X'_s))\, dB_s$$

is a mean 0 martingale.

$$\Rightarrow \mathbb{E}(|X_{t\wedge T_n} - X'_{t\wedge T_n}|^2) \leq (2K_n + K_n^2) \int_0^t \mathbb{E}(|X_{s\wedge T_n} - X'_{s\wedge T_n}|^2)\, ds$$

$$\Rightarrow f_n(t) \leq \underbrace{f_n(0)}_{=0} + C \int_0^t f_n(s)\, ds$$

By Gronwall's Lemma (f_n is bounded), thus $f_n(t) \leq f_n(0) e^{Ct} = 0$.

<u>Gronwall's Lemma</u> ($\rightarrow$ Example Sheet). Let $T > 0$ and let $f : [0,T] \rightarrow \mathbb{R}$ be a nonnegative bounded Borel function.

Assume

$$f(t) \leq a + b \int_0^t f(s)\, ds \quad \text{for all } t \leq T.$$

Then $f(t) \leq a e^{bt}$ for all $t \leq T$.

<u>Rk</u>. The proof of the theorem shows that solutions defined up to time T must agree up to time T.

5.2. Strong existence for Lipschitz coefficients

<u>Thm.</u> Assume b and σ are globally Lipschitz, i.e., there is $K>0$ s.t. for all $x, y \in \mathbb{R}^d$, $t>0$,

$$|b(t,x) - b(t,y)| \leq K|x-y|$$
$$|\sigma(t,x) - \sigma(t,y)| \leq K|x-y|.$$

For any $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$, any $(\mathcal{F}_t)$-Brownian motion B, any $x \in \mathbb{R}^d$, there is a (unique) strong solution to $E_x(\sigma, b)$.

<u>Proof.</u> To simplify notation, we assume $d=m=1$. Define

$$F(X)_t = x + \int_0^t \sigma(s, X_s)\, dB_s + \int_0^t b(s, X_s)\, ds.$$

Given any $(\mathcal{F}_t)$ we will find $(\mathcal{F}_t)$-adapted X s.t. $F(X)=X$. Such a fixed point is a strong solution since we can choose $(\mathcal{F}_t)$ to be filtration induced by B. We use Picard iter.

Fix $T>0$. For X continuous, adapted, set

$$|||X|||_T = \mathbb{E}\Big(\sup_{t\in[0,T]} |X_t|^2\Big)^{1/2}.$$

Then $\mathcal{B} = \{X : \Omega \times [0,T] \to \mathbb{R} : |||X|||_T < \infty\}$ is a Banach space.

Claim: $|||F(X) - F(Y)|||_T^2 \leq (2T+8)K^2 \int_0^T |||X-Y|||_t^2\, dt$.

$$|||F(X)-F(Y)|||_T^2 \underset{(a+b)^2 \le 2a^2+2b^2}{\le} 2\underbrace{\mathbb{E}\left(\sup_{t\le T}\left|\int_0^t (b(s,X_s)-b(s,Y_s))\,ds\right|^2\right)}_{(A)}$$

$$+\,2\underbrace{\mathbb{E}\left(\sup_{t\le T}\left|\int_0^t (\sigma(s,X_s)-\sigma(s,Y_s))\,dB_s\right|^2\right)}_{(B)}.$$

where

$$(A) \overset{(CS)}{\le} T\,\mathbb{E}\left(\sup_{t\le T}\int_0^t |b(s,X_s)-b(s,Y_s)|^2\,ds\right) \le TK^2\int_0^T |||X-Y|||_t^2\,dt$$

$$(B) \le 4\,\mathbb{E}\left(\int_0^T |\sigma(s,X_s)-\sigma(s,Y_s)|^2\,ds\right) \le 4K^2\int_0^T |||X-Y|||_t^2\,dt$$

Doob's inequality: $\mathbb{E}\left(\sup_{t\le T}|M_t|^2\right) \le 4\,\mathbb{E}M_T^2 = 4\mathbb{E}\langle M\rangle_T$

Claim: $|||F(0)|||_T < \infty$.

$$F(0)_t = x + \int_0^t b(s,0)\,ds + \int_0^t \sigma(s,0)\,dB_s$$

$$\Rightarrow |||F(0)|||_T^2 \le 3\left(|x|^2 + \underbrace{|||\int_0^{\cdot} b(s,0)\,ds|||_T^2}_{(A')} + \underbrace{|||\int_0^{\cdot} \sigma(s,0)\,dB_s|||_T^2}_{(B')}\right) < \infty$$

where similarly

$$(A') \overset{(CS)}{\le} T\int_0^T b(s,0)^2\,ds < \infty$$

$$(B') \overset{(Doob)}{\le} 4\,\mathbb{E}\left(\int_0^T |\sigma(s,0)|^2\,ds\right) < \infty$$

To define a solution to $E_x(\sigma,b)$ for all t, let $X^0_t = 0$. $\forall t$.

Set $X^{i+1} = F(X^i)$.

$$\Rightarrow |||X^{i+1} - X^i|||_T^2 \leq C \int_0^T |||X^i - X^{i-1}|||_t^2 \, dt$$

$$\leq C^2 \int_0^T \int_0^t |||X^i - X^{i-1}|||_s^2 \, ds \, dt$$

$$\leq \cdots \leq \frac{(CT)^i}{i!} |||X^1 - X^0|||_T$$

$$\Rightarrow \sum_{i=1}^{\infty} |||X^i - X^{i-1}|||_T^2 < \infty \quad \forall T$$

$\Rightarrow X^i$ converges uniformly on $[0,T]$, a.s., $\forall T \Rightarrow F(X) = X$

By uniqueness, solutions defined up to different T must agree when both are defined. Hence we can extend them to $\mathbb{R}_+$.

Rk. Suppose that $\sigma(t,x) = 1$ for all (t,x). Then the SDE $E(\sigma,b)$ is

$$X_t = X_0 + \int_0^t b(s, X_s) \, ds + B_t.$$

The same proof shows that this integral equation can be solved for any continuous function B if one replaces $|||\cdot|||_T$ by $\|\cdot\|_T = \sup_{t \leq T} |X_t|$.

The following proposition provides a (rough) estimate on the dependence of the solution on the initial condition.

Prop. Under the assumptions of the theorem let X^x be the solution with initial condition $X_0 = x$. Then, for $p \geq 2$,

$$\mathbb{E}\left(\sup_{s\leq t} |X^x_s - X^y_s|^p\right) \leq C_p |x-y|^p \exp(C_p (t\wedge 1)^p t)$$

Proof. For simplicity $d=1$. Fix $x,y \in \mathbb{R}^d$ and let

$$T_n = \inf\{t \geq 0 : |X^x_t| > n \text{ or } |X^y_t| > n\}.$$

Since $|a+b+c|^p \leq 3^{p-1}(|a|^p + |b|^p + |c|^p)$,

$$\mathbb{E}\left(\sup_{s\leq t} |X^x_{s\wedge T_n} - X^y_{s\wedge T_n}|^p\right) \leq 3^{p-1}\Bigg[|x-y|^p + \underbrace{\mathbb{E}\left(\sup_{s\leq t}\left|\int_0^{s\wedge T_n} (\sigma(r,X^x_r) - \sigma(r,X^y_r))\,dB_r\right|^p\right)}_{(A)}$$

$$+ \underbrace{\mathbb{E}\left(\sup_{s\leq t}\left|\int_0^{s\wedge T_n} (b(r,X^x_r) - b(r,X^y_r))\,dr\right|^p\right)}_{(B)}\Bigg]$$

The Burkholder-Davis-Gundy inequality ($\rightarrow$ Example Sheet) states that

$$\mathbb{E}\left(\left(\sup_{s\leq t} M_s\right)^p\right) \leq C_p \mathbb{E}(\langle M\rangle_t^{p/2}).$$

Note that if $p=2$ this follows from Doob's L^2 inequality as before:

$$\mathbb{E}\left(\left(\sup_{s\leq t} M_s\right)^2\right) \leq 4\mathbb{E} M_t^2 = 4\mathbb{E}\langle M\rangle_t.$$

Hence

$$(A) \overset{(BDG)}{\leq} C_p \, \mathbb{E}\left(\left(\int_0^{t\wedge T_n} (\sigma(r,X_r^x)-\sigma(r,X_r^y))^2 \, dr\right)^{p/2}\right)$$

(use $(t^{1-\frac{2}{p}})^{\frac{p}{2}} = t^{\frac{p}{2}-1}$)

$$\overset{(\text{Hölder})}{\leq} C_p \, t^{\frac{p}{2}-1} \, \mathbb{E}\left(\int_0^{t\wedge T_n} (\sigma(r,X_r^x)-\sigma(r,X_r^y))^p \, dr\right)$$

$$\leq C_p \, t^{\frac{p}{2}-1} \, \mathbb{E}\left(\int_0^{t} |\sigma(r\wedge T_n, X_{r\wedge T_n}^x)-\sigma(r\wedge T_n, X_{r\wedge T_n}^y)|^p \, dr\right).$$

(Note also that if $\sigma(r,x)=1$ then $(A)=0$.)

Only using Hölder,

$$(B) \leq C_p \, t^{p-1} \, \mathbb{E}\left(\int_0^t |b(r\wedge T_n, X_{r\wedge T_n}^x) - b(r\wedge T_n, X_{r\wedge T_n}^y)|^p \, dr\right).$$

Using the Lipschitz assumption,

$$\underbrace{\mathbb{E}\left(\sup_{s\leq t} |X_{s\wedge T_n}^x - X_{s\wedge T_n}^y|^p\right)}_{f_n(s)} \leq 3^{p-1}|x-y|^p + \hat{C}_p (t\wedge 1)^p \int_0^t \underbrace{\mathbb{E}\left(\sup_{s\leq r} |X_{s\wedge T_n}^x - X_{s\wedge T_n}^y|^p\right)}_{f_n(r)} ds$$

Note that f_n is bounded for $n<\infty$. By Gronwall, thus

$$f_n(s) \leq 3^{p-1}|x-y|^p \exp\left(\hat{C}_p (t\wedge 1)^p t\right).$$

By Fatou, we can take the limit $n\to\infty$ and get the claim.

Strong solutions can be considered functions of Brownian motion in the following sense. We recall the (d-dimensional) Wiener space $(W^d, \mathcal{W}^d, P)$ where

$$W^d = C(\mathbb{R}_+, \mathbb{R}^d), \quad \mathcal{W} = \sigma(X_t^i : t \in \mathbb{R}_+, i = 1, \dots, d),$$

$X_t(w) = w(t)$ for $w \in W$ and P is probability measure on $(W^d, \mathcal{W}^d)$ s.t. (X_t) is a Brownian motion with $X_0 = 0$.

The space $C(\mathbb{R}_+, \mathbb{R}^d)$ can be given the topology of uniform convergence on compact intervals. This topology is induced by the metric

$$d(w, w') = \sum_{k=1}^{\infty} \alpha_k \Big(\sup_{s \le k} |w(s) - w'(s)| \wedge 1 \Big) \qquad (*)$$

for any sequence $(\alpha_k) \subset \mathbb{R}_+$ with $\sum_k \alpha_k = 1$. (or $\sum_k \alpha_k < \infty$)

<u>Rk</u>. This metric makes $C(\mathbb{R}_+, \mathbb{R}^d)$ a complete separable metric space (a so called Polish space).

<u>Thm</u>. Under the assumption of the previous theorem, for $x \in \mathbb{R}^d$, there exist maps

$$F_x : \underbrace{C(\mathbb{R}_+, \mathbb{R}^m)}_{W^m} \longrightarrow \underbrace{C(\mathbb{R}_+, \mathbb{R}^d)}_{W^d}$$

measurable w.r.t. the completion of $\mathcal{W}^m$ on W^m and w.r.t. $\mathcal{W}^d$ on W^d, such that

(i) $\forall t \geq 0$: $F_x(w)_t$ is a measurable function of $\sigma(w(s): s \leq t)$, P-almost surely

(ii) $\forall w \in C(\mathbb{R}_+, \mathbb{R}^m)$: $x \in \mathbb{R}^d \mapsto F_x(w) \in C(\mathbb{R}_+, \mathbb{R}^d)$ is continuous

(iii) $\forall x \in \mathbb{R}^d$, $\forall (\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ satisfying the usual conditions, every $(\mathcal{F}_t)$-Brownian motion $\hat{B}$ with $\hat{B}_0 = 0$, the unique solution to $E_x(\sigma, b)$ is $\hat{X}_t = F_x(\hat{B})_t$.

(iv) In the set-up of (iii), if U is $\mathcal{F}_0$-measurable, then $F_U(\hat{B})_t$ is the unique solution with $X_0 = U$.

<u>Proof</u>. For simplicity, again $d = m = 1$. Let

$$\mathcal{G}_t = \sigma(w(s): 0 \leq s \leq t) \vee \mathcal{N}, \quad \mathcal{G} = \mathcal{G}_\infty$$

where $\mathcal{N}$ are the P-null sets. Then by the last theorem applied to $(W, \mathcal{G}, (\mathcal{G}_t), P)$, $B_t(w) = w(t)$, there is a unique strong solution X^x to $E_x(\sigma, b)$.

By the last proposition and $(*)$ with (α_k) chosen appropriately,

$$\mathbb{E}\, d(X^x, X^y)^p \leq \mathbb{E}\left(\left(\sum_k \alpha_k \sup_{s \leq k} |X_s^x - X_s^y|\right)^p\right)$$

$$\overset{\text{(Jensen)}}{\leq} \sum_k \alpha_k\, \mathbb{E}\left(\sup_{s \leq k} |X_s^x - X_s^y|^p\right)$$

$$\leq C_p |x-y|^p \sum_k \alpha_k \exp(C_p k^{p+1}) \leq \hat{C}_p |x-y|^p. \qquad (\dagger)$$

A version of Kolmogorov's continuity criterion applies to processes with values in a complete metric space indexed by $\mathbb{R}^d$ if (†) holds with $p > d$. Applying this to $(X^x, x \in \mathbb{R}^d)$, there is a modification $(\tilde{X}^x, x \in \mathbb{R}^d)$ that is continuous. We set

$$F_x(w) = \tilde{X}^x(w) = (\tilde{X}^x_t(w))_{t \geq 0}.$$

This gives (ii).

For (i), $w \mapsto F_x(w)_t = \tilde{X}^x_t(w)$ is $\mathcal{G}_t$-measurable. Since $X^x = \tilde{X}^x$ P-a.s. this gives (i).

Next, we show (iii). Given $(\Omega, (\mathcal{F}_t), \mathbb{P})$ and $\hat{B}$, we set

$$\hat{X}_t = F_x(\hat{B})_t.$$

Since F maps into $C(\mathbb{R}_+, \mathbb{R}^d)$, $\hat{X}$ is continuous in t. Since $F(\hat{B})_t$ coincides almost surely with a measurable function of $(\hat{B}_r : 0 \leq r \leq t)$ and $(\mathcal{F}_t)$ is complete, it follows that $\hat{X}$ is adapted. By definition,

$$\tilde{X}_t = x + \int_0^t \sigma(s, \hat{X}_s)\, dB_s + \int_0^t b(s, \tilde{X}_s)\, ds$$

$$= x + \lim_{m \to \infty} \sum_{i=1}^{\lfloor 2^m t \rfloor} \sigma(s, \tilde{X}_{(i-1)2^{-m}})\left(B_{i2^{-m}} - B_{(i-1)2^{-m}}\right) + \int_0^\infty b(s, \tilde{X}_s)\, ds$$

in P-prob. and thus P-a.s. along a subsequence which we fix now

Since $\hat{X}^x(w) = F_x(w)$ thus (along this subsequence)

$$F_x(w)_t = x + \lim_{m\to\infty} \sum_{m=1}^{\lfloor 2^m t \rfloor} \sigma(s, F_x(w)_{(i-1)2^{-m}}) \left(w(i2^{-m}) - w((i-1)2^{-m}) \right)$$
$$+ \int_0^t b(s, F(w)_s)\, ds$$

for P-a.e. $w \in W^m$

Since $\hat{B}$ has law P on W^m we can substitute $w = \hat{B}$ and then revert the approximation of the stochastic integral to get

$$\hat{X}_t = x + \int_0^t \sigma(s, \hat{X}_s)\, d\hat{B}_s + \int_0^t b(s, \hat{X}_s)\, ds$$

as claimed.

(iv) omitted. (The proof is similar.)

<u>Cor.</u> The solutions to $E_x(\sigma, b)$ can be constructed for all $x \in \mathbb{R}^d$ simultaneously such that a.s. X^x is continuous in the initial condition x.

5.3. Examples of SDEs

The Ornstein-Uhlenbeck process. Let $\lambda > 0$. The Ornstein-Uhlenbeck process is the (unique) solution to

$$dX_t = -\lambda X_t \, dt + dB_t.$$

It's coefficients are Lipschitz, so clearly it has a unique solution. It can be solved explicitly.

Apply Itô's formula to $e^{\lambda t} X_t$:

$$d(e^{\lambda t} X_t) = e^{\lambda t} dX_t + \lambda e^{\lambda t} X_t \, dt = e^{\lambda t} dB_t$$

$$\Leftrightarrow e^{\lambda t} X_t - X_0 = \int_0^t e^{\lambda s} dB_s$$

$$\Leftrightarrow X_t = e^{-\lambda t} X_0 + \int_0^t e^{-\lambda(t-s)} dB_s$$

deterministic $\to$ Wiener integral

Fact. If $X_0 = x$ is fixed (or Gaussian), then (X_t) is a Gaussian process, i.e., $(X_{t_i})_{i=1}^n$ is jointly Gaussian for all $0 = t_0 < \dots < t_n$.

Proof. Exercise.

Fact. $\mathbb{E} X_t = e^{-\lambda t} x$, $\quad \operatorname{cov}(X_t, X_s) = \frac{1}{2\lambda}\left(e^{-\lambda|t-s|} - e^{-\lambda|t+s|}\right)$.

Proof. Clearly,

$$\mathbb{E}X_t = e^{-\lambda t}\,\mathbb{E}X_0 + \underbrace{\mathbb{E}\int_0^t e^{-\lambda(t-s)}\,dB_s}_{=0} = e^{-\lambda t}x.$$

By Itô's isometry,

$$\begin{aligned}
\operatorname{cov}(X_t, X_s) &= \mathbb{E}\big((X_t - \mathbb{E}X_t)(X_s - \mathbb{E}X_s)\big) \\
&= \mathbb{E}\left(\left(\int_0^t e^{-\lambda(t-u)}\,dB_u\right)\left(\int_0^s e^{-\lambda(s-u)}\,dB_u\right)\right) \\
&= \int_0^\infty \left(1_{u<t}\, e^{-\lambda(t-u)}\right)\left(1_{u<s}\, e^{-\lambda(s-u)}\right) du \\
&= e^{-\lambda(t+s)} \int_0^{s\wedge t} e^{2\lambda u}\,du = \frac{1}{2\lambda} e^{-\lambda(t+s)}\left(e^{2\lambda(s\wedge t)} - 1\right)
\end{aligned}$$

Cor. $X_t \sim \mathcal{N}(\underbrace{e^{-\lambda t}x}_{\to 0}, \underbrace{\frac{1-e^{-2\lambda t}}{2\lambda}}_{\to \frac{1}{2\lambda}})$ for every $t \geq 0$, $X_0 = x \in \mathbb{R}$ as $t \to \infty$

This suggests that the distribution $\mathcal{N}(0, \frac{1}{2\lambda})$ is invariant. And indeed it is easy to check this.

Fact. If $X_0 \sim \mathcal{N}(0, \frac{1}{2\lambda})$, then $X_t \sim \mathcal{N}(0, \frac{1}{2\lambda})$ for all $t > 0$, and X_t is a stationary Gaussian process with

$$\operatorname{cov}(X_t; X_s) = \operatorname{cov}(X_0, X_{|t-s|}) = \frac{1}{2\lambda} e^{-\lambda|t-s|}.$$

Example. For $\beta \in C(\mathbb{R}_+, \mathbb{R})$, define $F_x(\beta)$ by

$$[F_x(\beta)](t) = x \exp\left(\sigma \beta(t) + (\mu - \frac{\sigma^2}{2})t\right).$$

If B is a standard BM with B_0, then $X_t = F_x(B)_t$ satisfies

$$dX_t = \sigma X_t \, dB_t + \mu X_t \, dt, \quad X_0 = x. \qquad (*)$$

On the other hand, if we choose β to be any smooth path, then $X_t = F_x(\beta)_t$ satisfies the ODE

$$dX_t = \sigma X_t \, d\beta_t + \left(\mu - \frac{\sigma^2}{2}\right) dt.$$

Thus the Itô map F satisfies the 'wrong' equation on smooth paths.

The process solving $(*)$ is called Geometric Brownian Motion.

5.4. Local solutions

As for ODEs, solutions to SDEs do not always exist for all times. For SDEs the explosion time is random.

Prop. (Local Itô formula). Let $X=(X^1,\dots,X^d)$ be continuous semimartingales. Let $U\subseteq\mathbb{R}^d$ be open and let $f:U\to\mathbb{R}^d$ be C^2. Set $T=\inf\{t\geq 0: X_t\notin U\}$. Then for all $t<T$,

$$f(X_t)=f(X_0)+\sum_{i=1}^{d}\frac{\partial f}{\partial x^i}(X_s)\,dX^i_s+\frac{1}{2}\sum_{i,j=1}^{d}\frac{\partial^2 f}{\partial x^i\partial x^j}(X_s)\,d\langle X^i,X^j\rangle_s.$$

Proof sketch. Apply Itô's formula to X^{T_n} where

$$T_n=\inf\{t\geq 0:\ \mathrm{dist}(X_t,U^c)\leq\tfrac{1}{n}\}.$$

Observe that $T_n\uparrow T$ as $n\to\infty$.

Example Let B be a standard Brownian motion with $B_0=1$ in dimension 1. Taking $U=(0,\infty)$, $f(x)=\sqrt{x}$ gives

$$\sqrt{B_t}=1+\frac{1}{2}\int_0^t B_s^{-1/2}\,ds-\frac{1}{8}\int_0^t B_s^{-3/2}\,ds$$

for $t<T=\inf\{t\geq 0: B_t=0\}$.

Thm. Let $U \subset \mathbb{R}^d$ be open and $b : \mathbb{R}_+ \times U \to \mathbb{R}^d$ and $\sigma : \mathbb{R}_+ \times U \to \mathbb{R}^{d \times m}$ be locally Lipschitz continuous. Then for every $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ and B, every $x \in U$, there exists a stopping time T such that, for $t < T$,

$$X_t = x + \int_0^t b(s, X_s)\, ds + \int_0^t \sigma(s, X_s)\, dB_s,$$

and for all $K \subset U$ compact, $\sup\{t < T : X_t \in K\} < T$.
T is called the <u>explosion time</u>.

<u>Proof</u>. Fix $K_n \subset U$ compact such that $K_{n+1} \supseteq K_n$ and $\bigcup_n K_n = U$. One can find b_n and σ_n defined on all of $\mathbb{R}^d$ such that $b_n|_{K_n} = b|_{K_n}$ and $\sigma_n|_{K_n} = \sigma|_{K_n}$ and such that b_n and σ_n are globally Lipschitz continuous. Hence there are unique solutions X^n to $E_x(\sigma_n, b_n)$. Let

$$T_n = \inf\{t \geq 0 : X_t^n \notin K_n\}.$$

By uniqueness, X^{n+1} also solves $E_x(\sigma_n, b_n)$ up to time T_n. Thus $X_t^{n+1} = X_t^n$ for $t < T_n$ and we can define X_t for $t < T := \sup_n T_n$ by requiring that $X_t = X_t^n$ for $t < T_n$.

Claim: Let $K \subset U$ be compact. Then on $\{T < \infty\}$ a.s.

$$\sup\{t < T : X_t \in K\} < T.$$

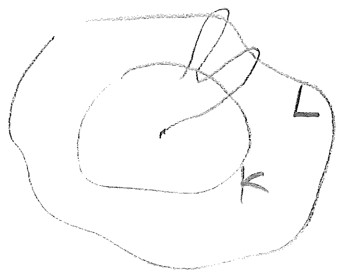
Let L be another compact set s.t. $K \subset \mathring{L} \subset L \subset U$.

Let $\varphi : U \to \mathbb{R}$ be C^∞ s.t. $\varphi|_K = 1$ and $\varphi|_{\mathring{L}^c} = 0$.

Let $R_1 = \inf\{t \geq 0 : X_t \notin L\}$

$S_n = \inf\{t \geq R_n : X_t \in K\} \wedge T$

$R_n = \inf\{t \geq S_{n-1} : X_t \notin L\}$.



Let N be the number of crossings of X from $\mathring{L}^c$ to K.

Then on $\{T \leq t, N \geq n\}$,

$$n = \sum_{k=1}^{n} \left(\varphi(X_{S_k}) - \varphi(X_{R_k})\right)$$

$$= \int_0^t \sum_{k=1}^{n} 1_{(R_k, S_k]}(s) \left(\varphi'(X_s)\, dX_s + \tfrac{1}{2}\varphi''(X_s)\, d\langle X\rangle_s\right)$$

$$= \int_0^t \left(H_s^n \, dB_s + \tilde{H}_s^n \, ds\right) = Z_t^n$$

with H^n and $\tilde{H}^n$ predictable and bounded independently of n.

$$\Rightarrow n^2 1_{\{T \leq t, N \geq n\}} \leq (Z_t^n)^2$$

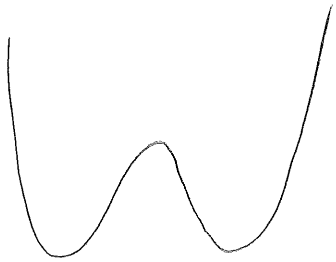
$$\Rightarrow P(T \leq t, N \geq n) \leq \frac{1}{n^2} E((Z_t^n)^2) \leq \frac{C(t)}{n^2}$$

$$\Rightarrow P(T \leq t, N = \infty) = 0$$

$\Rightarrow P(T < \infty, N = \infty) = 0$ which implies the claim.

<u>Example</u>. Consider the SDEs

$$dX_t^i = -\nabla_i H(X_t)\,dt + dB_t\,; \quad X_0 = x$$



Assume that there are $a \ge 0$, $b \ge 0$ s.t.

$$x \cdot \nabla H(x) \ge -a|x|^2 - b.$$

Then the SDE has a global solution, i.e., $T = \infty$ a.s.

<u>Proof</u>. Let $T_n = \inf\{t \ge 0 : |X_t|^2 > n\}$. Then by Itô's formula

$$\begin{aligned}
\mathbb{E}|X_{t\wedge T_n}|^2 &= \mathbb{E}|X_0|^2 - \mathbb{E}\int_0^{t\wedge T_n} X_s \cdot \nabla H(X_s)\,ds + t\wedge T_n \\
&\le \mathbb{E}|X_0|^2 + a\,\mathbb{E}\int_0^{t\wedge T_n} |X_s|^2\,ds + (1+b)(t\wedge T_n) \\
&\le \mathbb{E}|X_0|^2 + (1+b)t + a\int_0^t \mathbb{E}|X_{s\wedge T_n}|^2\,ds.
\end{aligned}$$

By Gronwall,

$$\mathbb{E}|X_{t\wedge T_n}|^2 \le \left(\mathbb{E}|X_0|^2 + (1+b)t\right)e^{at}.$$

Since $|X_{t\wedge T_n}|^2 \to \infty$ on $\{T < \infty\}$ it follows that $\mathbb{P}(T < \infty) = 0$.

6. Applications to PDEs and Markov processes

6.1. Probabilistic representations of solutions to PDEs

Exercise. Let $b: \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma: \mathbb{R}^d \to \mathbb{R}^{d\times m}$ be (locally) bounded Borel functions, and let $x \in \mathbb{R}^d$. Assume that X is a solution to $E_x(\sigma, b)$. Then, for every $f \in C^1(\mathbb{R}_+) \otimes C^2(\mathbb{R}^d)$,

$$M_t^f = f(t, X_t) - f(0, X_0) - \int_0^t \left(\frac{\partial}{\partial s} + L\right) f(s, X_s)\, ds$$

is a continuous local martingale where

$$Lf(y) = \frac{1}{2}\sum_{i,j=1}^d a_{ij}(y) \frac{\partial^2 f}{\partial y^i \partial y^j} + \sum_{i=1}^d b_i(y) \frac{\partial f}{\partial y^i}, \quad a(y) = \sigma(y)\sigma(y)^T \in \mathbb{R}^{d\times d}$$

Defn. L is called the (infinitesimal) generator of X.

Examples.

- $dX = dB$ (Brownian motion) $\longleftrightarrow L = \frac{1}{2}\Delta = \frac{1}{2}\sum_{i=1}^d \frac{\partial^2}{(\partial y^i)^2}$
- $dX = -X\,dt + dB$ (Ornstein-Uhlenbeck process)

  $\leftrightarrow L = \frac{1}{2}\Delta - x\cdot\nabla$

Rk. Let L be as above. A probility measure on the Wiener space is a martingale solution to $E(\sigma, b)$ if M^f is a continuous local martingale for all $f \in C^2(\mathbb{R}^d)$. $\lrcorner$

Dirichlet-Poisson problem. Let $U \subset \mathbb{R}^d$, $U \neq \emptyset$ be open and bounded. Given $f \in C(\overline{U})$, $g \in C(\partial U)$, find $u \in C^2(\overline{U}) = C^2(U) \cap C(\overline{U})$, s.t.

$$\text{(DP)} \quad \begin{cases} -Lu(x) = f(x) & \text{for } x \in U \\ u(x) = g(x) & \text{for } x \in \partial U. \end{cases}$$

Dirichlet problem: $f = 0$
Poisson problem: $g = 0$

Defn. $a : \overline{U} \to \mathbb{R}^{d \times d}$ is uniformly elliptic if there is $c > 0$ s.t.

$$\xi^T a(x) \xi \geq c|\xi|^2 \text{ for all } \xi \in \mathbb{R}^d,\ x \in \overline{U}.$$

Thm. ($\to$ Evans, Gilbarg-Trudinger, ...). Assume that U has a smooth boundary (or satisfies the exterior sphere condition), that a and b are Hölder continuous, and that a is uniformly elliptic. Then for every Hölder continuous $f : \overline{U} \to \mathbb{R}$ and every continuous $g : \partial U \to \mathbb{R}$, the Dirichlet-Poisson problem has a solution.

Thm. Let $U \subset \mathbb{R}^d$ be open, bounded, $U \neq \emptyset$, let σ and b be bounded and measurable, assume $a = \sigma\sigma^T$ is uniformly elliptic. Let u be a solution to (DP). Let $x \in U$, and let X be a solution to $E_x(\sigma, b)$. Let $T_U = \inf\{t \geq 0 : X_t \notin U\}$.
Then $\mathbb{E}\, T_U < \infty$ and

$$u(x) = \mathbb{E}_x\left(u(X_{T_U}) - \int_0^{T_U} Lu(X_s)\, ds\right) = \mathbb{E}_x\left(g(X_{T_U}) + \int_0^{T_U} f(X_s)\, ds\right).$$

(sometimes called Dynkin's formula)

Proof. Let

$$T_n = \inf\{t \geq 0 : X_t \notin U_n\}, \quad U_n = \{x \in U : \mathrm{dist}(x, \partial U) > \tfrac{1}{n}\}.$$

There are $u_n \in C_b^2(\mathbb{R}^d)$ s.t. $u|_{U_n} = u_n|_{U_n}$. Then

$$M_t^n = (M^{u_n})_t^{T_n} = u_n(X_{t \wedge T_n}) - u_n(X_0) - \int_0^{t \wedge T_n} Lu_n(X_s)\, ds$$

is a local mart., bounded for $t \leq t_0$, so a martingale.

$$\Rightarrow u(x) \underset{\substack{\uparrow \\ \text{for } x \in U \text{ and} \\ n \text{ large enough that } x \in U_n}}{=} u_n(x) = \mathbb{E}\Big(\underbrace{u_n(X_{t \wedge T_n})}_{= u(X_{t \wedge T_n})} - \int_0^{t \wedge T_n} \underbrace{Lu_n(X_s)}_{-f(X_s)}\, ds\Big)$$

To take the limit $t \wedge T_n \to T_U$, we will need $\mathbb{E}\, T_U < \infty$.

To see $\mathbb{E}\, T_U < \infty$, let v be the solution to (DP) with $f(x) = 1\ \forall x$, $g(x) = 0\ \forall x$.

Then

$$\mathbb{E}(t\wedge T_n) = \mathbb{E}\Big(\int_0^{t\wedge T_n} \underbrace{-Lv(X_s)}_{=1}\, ds\Big) = v(x) - \mathbb{E}\big(v(X_{t\wedge T_n})\big) \le 2\|v\|_\infty .$$

By monotone convergence and since $T_n \uparrow T$, thus

$$\mathbb{E}(T_U) = \lim_{t\to\infty} \lim_{n\to\infty} \mathbb{E}(t\wedge T_n) \le 2\|v\|_\infty < \infty .$$

Claim: $u(x) = \mathbb{E}\big(u(X_{T_U}) + \int_0^{T_U} f(X_s)\, ds\big)$.

Since $t\wedge T_n \uparrow T_U$ as $n\to\infty$, $t\to\infty$, and since

$$\mathbb{E}\Big(\int_0^{T_U} |f(X_s)|\, ds\Big) \le \|f\|_\infty\, \mathbb{E}T_U < \infty ,$$

the DCT implies

$$\mathbb{E}\Big(\int_0^{t\wedge T_n} f(X_s)\, ds\Big) \longrightarrow \mathbb{E}\Big(\int_0^{T_U} f(X_s)\, ds\Big).$$

Since u is continuous on $\bar{U}$, also by DCT

$$\mathbb{E}\big(u(X_{t\wedge T_n})\big) \longrightarrow \mathbb{E}\big(u(X_{T_U})\big).$$

Cauchy Problem. Given $f \in C_b^2(\mathbb{R}^d)$, find $u \in C^1(\mathbb{R}_+) \otimes C^2(\mathbb{R}^d)$ s.t.

$$(CP) \begin{cases} \frac{\partial}{\partial t} u = Lu & \text{on } (0,\infty) \times \mathbb{R}^d \\ u(0,\cdot) = f & \text{on } \mathbb{R}^d \end{cases}$$

Thm ($\to$ Evans, Gilbarg-Trudinger, ...). For every $f \in C_b^2(\mathbb{R}^d)$, there exists a solution to (CP).

Thm. Let u be a solution to (CP). Then for any solution X to $E_x(\sigma, b)$, $x \in \mathbb{R}^d$, $0 \leq s < t$,

$$\mathbb{E}(f(X_t) \mid \mathcal{F}_s) = u(t-s, X_s).$$

In particular, $\mathbb{E}_x(f(X_t)) = u(t,x)$.

Proof. Let $g(s,x) = u(t-s,x)$. Then

$$\Rightarrow \left(\frac{\partial}{\partial s} + L\right) g(s,x) = 0$$

$$\Rightarrow g(s, X_s) - g(s,x) \text{ is a martingale}$$

$$\Rightarrow u(t-s, X_s) = g(s, X_s) = \mathbb{E}(g(t, X_t) \mid \mathcal{F}_s) = \mathbb{E}(f(X_s) \mid \mathcal{F}_s).$$

Thm (Feynman–Kac formula). Let $f \in C_b^2(\mathbb{R}^d)$, $V \in C_b(\mathbb{R}^d)$, and suppose that $u : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$ satisfies

$$\begin{cases} \frac{\partial u}{\partial t} = Lu + Vu & \text{on } \mathbb{R}_+ \times \mathbb{R}^d \\ u(0,\cdot) = f \end{cases}$$

(pointwise multiplication)

Let X be a solution to $E_x(\sigma, b)$ for some $x \in \mathbb{R}^d$. Then for all $t \geq 0$,

$$u(t,x) = \mathbb{E}_x\left(f(X_t) \exp\left(\int_0^t V(X_s)\, ds \right)\right).$$

Proof. Let

$$E_t = \exp\left(\int_0^t V(X_s)\, ds \right).$$

For $s < t$, set $M_s = u(t-s, X_s) E_s$.

$$\Rightarrow dM_s = -\frac{\partial}{\partial t} u(t-s, X_s) E_s ds + \nabla u(t-s, X_s) E_s \sigma_t dB_t + Lu(t-s, X_s) E_s ds$$
$$+ u(t-s, X_s) V(X_s) E_s\, ds$$

$$= \underbrace{\left(-\frac{\partial}{\partial t} + L + V(X_s)\right) u(t-s, X_s)}_{=0 \text{ by assumption}} E_s\, ds + d(\text{mart.})$$

Thus M_s is a cont. local martingale on $[0,t]$. By assumption, M is also bounded, so a martingale.

$$\Rightarrow u(t,x) = M_0 = \mathbb{E}_x M_t = \mathbb{E}_x u(0, X_t) E_t = \mathbb{E}_x f(X_t) E_t.$$

6.2. Markov property

Let $B(\mathbb{R}^d)$ be the Banach space of bounded Borel functions on $\mathbb{R}^d$ with $\|f\| = \sup_{x\in\mathbb{R}^d} |f(x)|$ for $f \in B(\mathbb{R}^d)$.

Defn. (i) A collection of bounded linear operators Q_t on $B(\mathbb{R}^d)$ is a transition semigroup if $Q_t f \geq 0$ if $f \geq 0$ (pointwise), $Q_t 1 = 1$, $\|Q_t\| \leq 1$, and

$$Q_{t+s} = Q_t Q_s \text{ for all } t, s \geq 0.$$

(ii) An $(\mathcal{F}_t)$-adapted process X is a Markov process with transition semigroup (Q_t) if

$$\mathbb{E}(f(X_{s+t}) \mid \mathcal{F}_s) = Q_t f(X_s) \quad \forall s, t \geq 0,\ f \in B(\mathbb{R}^d).$$

Thm. Let $b: \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma: \mathbb{R}^d \to \mathbb{R}^{d\times m}$ be Lipschitz. Assume X is a solution to $E(\sigma, b)$ on some $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$. Then $X = (X_t)_{t\geq 0}$ is a Markov process with semigroup

$$Q_t f(x) = \mathbb{E}(f(X_t^x)) = \int f(F_x(w)_t)\, \mathbb{P}^m(dw)$$

where X^x is an arbitrary solution to $E_x(\sigma, b)$, F is the solution map on the Wiener space and $\mathbb{P}^m$ is the Wiener measure.

Proof. Let X be a solution to $E(\sigma, b)$.

Claim: $E(f(X_{t+s}) | \mathcal{F}_s) = Q_t f(X_s)$.

$$X_t = X_0 + \int_0^t \sigma(X_u)\, dB_u + \int_0^t b(X_u)\, du \quad \forall t \geq 0$$

$$\Rightarrow X_{t+s} = X_s + \int_s^{s+t} \sigma(X_u)\, dB_u + \int_s^{s+t} b(X_u)\, du \quad \forall t, s \geq 0.$$

Set $X'_t = X_{s+t}$, $\mathcal{F}'_t = \mathcal{F}_{s+t}$, $B'_t = B_{s+t} - B_s$.

Then $(\Omega, \mathcal{F}, (\mathcal{F}'_t), P)$ is another filtered probability space satisfying the usual conditions and B' another Brownian motion. We have

$$X'_t = X'_0 + \int_0^t \sigma(X'_u)\, dB'_u + \int_0^t b(X'_u)\, du$$

(For $\int_s^{s+t} \sigma(X_u) dB_u = \int_0^t \sigma(X'_u) dB'_u$ use approximation results.)

Thus X' solves $E(\sigma, b)$ with $X'_0 = X_s$. By the theorem on p. 91', therefore $X' = F_{X_s}(B')$ a.s.

$$\Rightarrow \mathbb{E}(f(X_{s+t}) | \mathcal{F}_s) = \mathbb{E}(f(X'_t) | \mathcal{F}_s) = \mathbb{E}(f(F_{X_s}(B')_t | \mathcal{F}_s)$$

B' is independent of X_s $\quad = \int f(F_{X_s}(w)_t)\, P^m(dw) = Q_t f(X_s)$.

Also,

$$Q_{t+s} f(x) = \mathbb{E}(f(X^x_{t+s})) = \mathbb{E}(\mathbb{E}(f(X^x_{t+s}) | \mathcal{F}_s)) = \mathbb{E} Q_t f(X^x_s) = Q_s Q_t f(x).$$

*

Defn. Let Q_t be the transition semigroup.

(i) A probability measure μ on $\mathbb{R}^d$ is invariant under (Q_t) if

$$\int Q_t f(x)\,\mu(dx) = \int f(x)\,\mu(dx) \quad \forall f \in B(\mathbb{R}^d).$$

(ii) A probability measure μ is reversible w.r.t. (Q_t) if

$$\int g(x)\,Q_t f(x)\,\mu(dx) = \int Q_t g(x)\, f(x)\,\mu(dx) \quad \forall f,g \in B(\mathbb{R}^d)$$

Fact. reversible $\Rightarrow$ invariant

Proof. Take $g=1$ and use $Q_t 1 = 1$.

Example. Consider the transition semigroup associated to the SDE

$$dX_t = -\tfrac{1}{2}\nabla H(X_t)\,dt + dB_t \qquad (*)$$

with suitable assumptions on H. Then the measure $\mu(dx) = \frac{1}{Z} e^{-H(x)}dx$, $Z = \int e^{-H(x)}dx$ is reversible.

Lemma. Assume that the explosion time for $(*)$ is infinite. Then for $f: C([0,T],\mathbb{R}^d) \to \mathbb{R}$,

$$\mathbb{E}\left(f(X|_{[0,T]})\right) = \mathbb{E}^{BM}\left(f(X|_{[0,T]}) \exp\left(\tfrac{1}{2}H(X_0) - \tfrac{1}{2}H(X_T) - \int_0^T \left(\tfrac{1}{8}|\nabla H|^2 - \tfrac{1}{4}\Delta H\right)(X_s)\,ds\right)\right).$$

law under which X is a Brownian motion with same initial cond.

$\Downarrow$

<u>Proof</u>. Let X be a standard Brownian motion under $\mathbb{E}^{BM}$.
Assume first that $|\nabla H|$ is bounded. Then

Let
$$L_t = -\frac{1}{2}\int_0^{t\wedge T} \nabla H(X_s)\, dX_s$$
$$\Rightarrow \langle L\rangle_{t\wedge T} = \frac{1}{4}\int_0^{t\wedge T} |\nabla H(X_s)|^2\, ds \Rightarrow \langle L\rangle_T \le CT.$$
$\Rightarrow \mathcal{E}(L)$ is UI martingale.

By Girsanov's Theorem,
$$X_t - \langle X, L\rangle_t = X_t + \int_0^t \nabla H(X_s)\, ds$$
is a standard BM under $\mathbb{E}(\cdot) = \mathbb{E}^{BM}((\cdot)\,\mathcal{E}(L)_\infty)$.
Thus X is a solution to $(*)$ under $\mathbb{E}$.

For bounded $|\nabla H|$ the claim follows since
$$\mathcal{E}(L)_\infty = \exp\left(-\int_0^T \frac{1}{2}\nabla H(X_s)\cdot dX_s - \frac{1}{8}\int_0^T |\nabla H(X_s)|^2\, ds\right)$$
$$\underset{\text{Itô}}{=} \exp\left(\frac{1}{2}H(X_0) - \frac{1}{2}H(X_T) + \frac{1}{4}\int_0^T \Delta H(X_s)\, ds - \frac{1}{8}\int_0^T |\nabla H(X_s)|^2\, ds\right)$$

The case of general H follows by approximation.

Sketch of proof of example.

$$\int Q_T f(x)\, g(x)\, e^{-H(x)}\, dx = \int \mathbb{E}\big(f(X_T^x)\big)\, g(x)\, e^{-H(x)}\, dx$$

$$= \int \underset{\substack{\uparrow \\ X \text{ is BM} \\ \text{with } X_0 = 0}}{\mathbb{E}^{BM}} \Big(f(x+X_T)\, g(x+X_0) \exp\Big(-\tfrac{1}{2}H(x+X_0) - \tfrac{1}{2}H(x+X_T) - \int_0^T \big(\tfrac{1}{8}|\nabla H|^2 - \tfrac{1}{4}\Delta H\big)(x+X_t)\, dt\Big)\Big) dx$$

$$= \int \mathbb{E}^{BM}\Big(f(x+\tilde X_0)\, g(x+\tilde X_T) \exp\Big(-\tfrac{1}{2}H(x+\tilde X_0) - \tfrac{1}{2}H(x+\tilde X_T) - \int_0^T \big(\tfrac{1}{8}|\nabla H|^2 - \tfrac{1}{4}\Delta H\big)(x+\tilde X_T)\, dt\Big)\Big) dt$$

where $\tilde X_t = X_{T-t}$

Since $\int \mathbb{E} F(x+X)\, dx = \int \mathbb{E} F(x+\tilde X)\, dx$ for any $F : C([0,T], \mathbb{R}^d) \to \mathbb{R}$, which can be checked by computing f.d. distributions, the claim follows.

—

Now many interesting questions can be studied. Does the law of X_t converge to the invariant measure μ for general X_0, etc.