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Problems marked with (†) may be handed in for marking (CCA pidgeonhole G/H). Problems marked with (⋆) are additional questions

Problem 1. Suppose that M is a continuous local martingale with $M_0 = 0$. Show that, if $\mathbb{E}([M]_t) < \infty$ for all $t \geq 0$, then M is a true martingale. Show further that M is an L^2-bounded martingale if, and only if, $\mathbb{E}([M]_\infty) < \infty$.

Problem 2.

i. Suppose that M, N are independent continuous local martingales. Show that $[M, N]_t = 0$. In particular, if $B^{(1)}$ and $B^{(2)}$ are the coordinates of a standard Brownian motion in $\mathbb{R}^2$, this shows that $[B^{(1)}, B^{(2)}]_t = 0$ for all $t \geq 0$.

ii. Let B be a standard Brownian motion in $\mathbb{R}$ and let T be a stopping time which is a.s. not constant. Show that T is measurable with respect to the σ-algebras generated by both B^T and $B - B^T$, and conclude that the converse to the previous part is false.

Problem 3. (†) (Burkholder inequality) Fix $p \geq 2$ and let M be a continuous local martingale with $M_0 = 0$. Use Itô's formula, Doob's inequality, and Hölder's inequality to show that there exists a constant $C_p > 0$ such that

$$\mathbb{E}\left(\sup_{0 \leq s \leq t} |M_s|^p\right) \leq C_p \mathbb{E}\big([M]_t^{p/2}\big).$$

Problem 4. Suppose that $f\colon [0, \infty) \to \mathbb{R}$ is a continuous function. Show that if f has finite variation then it has zero quadratic variation. Conversely, show that if f has finite and positive quadratic variation then it must be of infinite variation.

Problem 5. Let B be a standard Brownian motion. Use Itô's formula to show that the following are martingales with respect to the filtration generated by B.

i. $X_t = \exp(\lambda^2 t/2)\sin(\lambda B_t)$

ii. $X_t = (B_t + t)\exp(-B_t - t/2)$

iii. $X_t = \exp(B_t - t/2)$

Problem 6. We recall that a real-valued process (X_t) is Gaussian if for any finite family $0 \leq t_1 < t_2 < \cdots < t_n < \infty$, the random vector $(X_{t_1}, \dots, X_{t_n})$ is Gaussian.

Let $h\colon [0, \infty) \to \mathbb{R}$ be a measurable function which is square-integrable when restricted to $[0, t]$ for each $t > 0$, and let B be a standard Brownian motion. Show that the process $H_t = \int_0^t h(s) dB_s$ is Gaussian, and compute its covariance.

Problem 7. Show that convergence in $(\mathcal{M}_c^2, \|\cdot\|)$ implies ucp convergence.

Problem 8. Show that the covariation $[\cdot,\cdot]$ is symmetric and bilinear. That is, if $M_1, M_2, M_3 \in \mathcal{M}_{c,loc}$ and $a \in \mathbb{R}$, then

$$[aM_1 + M_2, M_3] = a[M_1, M_3] + [M_2, M_3].$$

Problem 9. Let B be a standard Brownian motion and let

$$\widehat{B}_t = B_t - \int_0^t \frac{B_s}{s} ds.$$

i. Show that $\widehat{B}$ is not a martingale in the filtration generated by B.

ii. Show that $\widehat{B}$ is a continuous Gaussian process and identify its mean and covariance. Hence show that $\widehat{B}$ is a martingale in its own filtration.

You may find the following property of Gaussian random variables helpful: if X_n is a sequence of Gaussian random variables taking values in $\mathbb{R}^d$, and if $X_n \to X$ almost surely, then X is also Gaussian.

Problem 10. (†) Fix $d \geq 3$ and let B be a Brownian motion in $\mathbb{R}^d$ starting at $B_0 = \overline{x} = (x, 0, \ldots, 0) \in \mathbb{R}^d$ for some $x > 0$. Let $\|\cdot\|$ denote the Euclidean norm on $\mathbb{R}^d$. For each $a > 0$, let $\tau_a = \inf\{t > 0 : \|B_t\| = a\}$.

i. Let $D = \mathbb{R}^d \setminus \{0\}$ and let $h\colon D \to \mathbb{R}$ be defined by $h(x) = \|x\|^{2-d}$. Show that h is harmonic on D and that $M_t = \|B_t^{\tau_a}\|^{2-d}$ is a local martingale for all $a \geq 0$. For which values of x is M a true $\mathbb{P}_{\overline{x}}$-martingale?

ii. Use the previous part to show that for any $a < b$ such that $0 < a < x < b$,

$$\mathbb{P}_{\overline{x}}[\tau_a < \tau_b] = \frac{\phi(b) - \phi(x)}{\phi(b) - \phi(a)}$$

where $\phi(u) = u^{2-d}$. Conclude that if $x > a > 0$, then

$$\mathbb{P}_x[\tau_a < \infty] = (a/x)^{d-2}.$$

Problem 11.

i. Let $f\colon \mathbb{C} \to \mathbb{C}$ be analytic and let $Z_t = X_t + iY_t$ where (X, Y) is a Brownian motion in $\mathbb{R}^2$. Use Itô's formula to show that $M = f(Z)$ is a local martingale in $\mathbb{R}^2$. Show further that M is a time-change of Brownian motion in $\mathbb{R}^2$.

ii. Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and fix $z \in \mathbb{D}$. What is the hitting distribution for Z on ∂D in the case that $Z_0 = 0$? By applying a Möbius transformation $\mathbb{D} \to \mathbb{D}$ and using the previous part, determine the hitting distribution for Z on $\partial\mathbb{D}$.

Problem 12 (Liouville's Theorem in d = 2) Suppose that $u : \mathbb{R}^2 \to \mathbb{R}$ is bounded, and harmonic, and fix $x, y \in \mathbb{R}^2$. Let B be a Brownian motion; for $\epsilon > 0$, let

$$T_\epsilon = \inf\{t \geq 0 : |B_t - y| \leq \epsilon\}.$$

Show that

$$u(x) = \mathbb{E}_x\left(u(B_{T_\epsilon})\right).$$

Deduce that $u(x) = u(y)$, and conclude that u is constant.

Problem 13*. (Mean Value Property) Let $U \subset \mathbb{R}^d$ be an open set. We say that a function $u \in L^\infty_{\text{loc}}(U)$ satisfies the *mean value property* if, whenever $S(x, r) \subset U$, we have

$$u(x) = \int_{S(x,r)} u(y)\mu_{x,r}(dy) \qquad \text{(MVP)}$$

where we write $\mu_{x,r}$ for the uniform distribution on the sphere $S(x, r) = \partial B(x, r)$.

i. Suppose $u \in C^2(U)$ is harmonic. Show that u satisfies (MVP).

ii. Suppose, conversely, that u satisfies (MVP). For any compact subset $K \subset U$, express $u|_K$ as a convolution, and deduce that $u \in C^\infty(U)$.

iii. Suppose u satisfies (MVP). Fix $x \in U$ and $r > 0$ such that $\overline{B(x, r)} \subset U$. Let B be a Brownian Motion started at x, and let $\tau_r = \inf\{t > 0 : \|x - B_t\| = r\}$. Show that

$$\forall t \geq 0, \qquad \mathbb{E}\left(\int_0^{t\wedge\tau_r} \triangle u(B_s)ds\right) = 0.$$

Deduce that u is harmonic. Hence (MVP) is an equivalent characterisation of harmonic functions.

Problem 14*. (Liouville's Theorem) Let $d \geq 3$, and suppose that $u : \mathbb{R}^d \to \mathbb{R}$ is bounded and harmonic. Let B be a Brownian motion starting at 0.

i. Show that $M_t = u(B_t)$ is a bounded martingale. Conclude that M_t converges, almost surely and in L^1, to a random variable M_∞.

ii. Recall Blumenthal's $0-1$ law. Show that the *tail σ-algebra*

$$\tau = \cap_{t \geq 0} \sigma(B_s : s \geq t)$$

contains only events of probability 0 and 1, and deduce that M_∞ is almost surely constant.

iii. Using the relationship between M_∞ and M_1, show that M_1 is almost surely constant, and conclude that u is constant.

Problem 15*. (Winding Numbers of Planar Brownian Motion)

i. Let X, Y be independent Brownian motions in one dimension, starting at 0, and for $x > 0$, let τ_x be the hitting time $\tau_x = \inf\{t \geq 0 : X_t = x\}$. Find the distribution of Y_{τ_x}.

ii. Let Z be a 2−dimensional Brownian motion, started at $(\epsilon, 0)$, and let τ be the hitting time

$$\tau = \inf\{t \geq 0 : |Z_t| = 1\}.$$

Let W_ϵ be the number of windings of Z around 0 before time τ; that is, every time Z completes a clockwise circuit of the origin, increase W_ϵ by 1, and similarly decrease W_ϵ by 1 for every counterclockwise circuit. Show that $\frac{2\pi W_\epsilon}{\log \epsilon}$ converges in distribution as $\epsilon \to 0$, and identify the limit.