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Problems marked with (†) may be handed in for marking (CCA pidgeonhole G/H). Problems marked with (⋆) are additional questions

Problem 1. Let $f\colon [0,\infty) \to \mathbb{R}$ be absolutely continuous, in the sense that

$$f(t) = f(0) + \int_0^t f'(s)ds \quad \text{for all} \quad t \geq 0$$

for an integrable function f'. Let $v_f(0,t)$ be the total variation of f on $(0,t]$. Show that

$$v_f(0,t) = \int_0^t |f'(s)|\, ds.$$

Problem 2. Let $f, g\colon [0,\infty) \to \mathbb{R}$ be bounded and measurable, and let $a\colon [0,\infty) \to \mathbb{R}$ be continuous and of finite variation. Show that

$$f \cdot (g \cdot a) = (fg) \cdot a$$

where $\cdot$ denotes the Lebesgue-Stieltjes integral.

Problem 3.

i. Suppose that $f : [0,T] \to \mathbb{R}$ is càdlàg and of bounded variation, and let $v_f(0,t)$ be its total variation on $(0,t]$. Show that, if $0 \leq s \leq t \leq T$, then

$$v_f(0,t) - v_f(0,s) = \sup\left\{\sum_{i=1}^n |f(u_i) - f(u_{i-1})| : n \in \mathbb{N}, s = u_0 \leq u_1 \leq \ldots \leq u_n = t\right\}. \tag{1}$$

ii. Using (1), show that v is càdlàg on $[0,T]$.

Problem 4. Let H be a previsible process. Let $\mathcal{F}_{t^-} = \sigma(\mathcal{F}_s : s < t)$. Show that H_t is $\mathcal{F}_{t-}$-measurable, for any $t > 0$.

Problem 5. Let $(\mathcal{F}_t)_{t\geq 0}$ be a filtration, let T be a stopping time, and let

$$\mathcal{F}_T = \{A \in \mathcal{F} : A \cap \{T \leq t\} \in \mathcal{F}_t \quad \forall t \geq 0\}.$$

i. Show that $\mathcal{F}_T$ is a σ-algebra.

ii. Show that T is $\mathcal{F}_T$-measurable.

iii. Suppose that X is a càdlàg, adapted process. Show that X_T is $\mathcal{F}_T$-measurable.

Problem 6. Let $(T_n)_{n\geq 1}$ denote a sequence of stopping time for a filtration $(\mathcal{F}_t)_{t\geq 0}$.

i. Show that $T^\star = \sup_n T_n$ is a stopping time for $(\mathcal{F}_t)_{t\geq 0}$.

ii. Show a random variable T is a stopping time for the filtration $\mathcal{F}_{t^+} = \bigcap_{s>t} \mathcal{F}_s$ if, and only if,

$$\{T < t\} \in \mathcal{F}_{t^+}$$

for all $t \geq 0$.

iii. Show that $T_\star = \inf_n T_n$ is a stopping time for $(\mathcal{F}_{t^+})_{t\geq 0}$.

Problem 7. (†) Let B be a standard Brownian motion.

i. Let $T = \inf\{t \geq 0 : B_t = 1\}$. Show that H defined by $H_t = \mathbf{1}\{T \geq t\}$ is previsible.

ii. Let

$$\operatorname{sgn}(x) = \begin{cases} -1 & \text{if} \quad x \le 0 \\ 1 & \text{if} \quad x > 0. \end{cases}$$

Show that $(\operatorname{sgn}(B_t))_{t\ge 0}$ is a previsible process which is neither left nor right continuous.

Problem 8. Let N be a Poisson process of rate 1, and let $X_t = N_t - t$ for $t \ge 0$. Show that X is of finite variation. Show that both X and $X_t^2 - t$ are martingales.

Problem 9. (Stochastic Calculus of a Total Variation Processes) Let T and ξ denote two independent random variables defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with

$$\mathbb{P}(T \le t) = t \ \text{ for } t \in [0,1], \qquad \mathbb{P}(\xi = 1) = \mathbb{P}(\xi = -1) = 1/2.$$

Define $X_t = \xi \mathbf{1}_{t \ge T}$ and $\mathcal{F}_t = \sigma(X_s : s \le t)$. Show that X is a martingale with respect to $(\mathcal{F}_t)_{t\in[0,1]}$, and that it is of finite variation. For bounded processes H, define pathwise

$$Y_t(\omega) := \int_{(0,t]} H_s(\omega) dX_s(\omega) \qquad \text{for all } \omega \in \Omega,$$

where the right-hand side is a Lebesgue-Stieltjes integral. Verify that, if H is a simple process

$$H_t = a_u \mathbf{1}_{t\in(u,v]}, \qquad a_u \in L^\infty(\mathcal{F}_u), \quad 0 \le u < v \le 1,$$

then (Y_t) is a martingale; use a monotone class argument to extend this to bounded, previsible H. What happens if we take $H = X$?

Problem 10. Suppose that $X \in L^1(\Omega, \mathcal{F}, \mathbb{P})$. Show that the family

$$\mathcal{X} = \{\mathbb{E}[X \,|\, \mathcal{G}] : \mathcal{G} \subseteq \mathcal{F} \quad \text{is a } \sigma\text{-algebra}\} \qquad \text{is UI.}$$

Problem 11. Let X be a continuous local martingale. Show that if

$$\mathbb{E}\Big(\sup_{0\le s\le t} |X_s| \Big) < \infty \qquad \forall t \ge 0$$

then X is a martingale.

Problem 12. (A silly martingale) Construct a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge 0}, \mathbb{P})$, a L^∞-bounded martingale $(M_t)_{t=0}^1$ and a stopping time T taking values in $[0,1]$, such that

$$\mathbb{E}(M_T) \neq \mathbb{E}(M_0).$$

Problem 13. (†) Let B be a standard Brownian motion and fix $t \ge 0$. For $n \ge 1$, let $\Delta_n = \{0 : t_0(n) < t_1(n) < \cdots < t_{m_n}(n) = t\}$ be a partition of $[0,t]$ such that

$$h_n = \max_{1\le i\le m_n} \big(t_i(n) - t_{i-1}(n)\big) \to 0 \quad \text{as} \quad n \to \infty.$$

Show that

$$[B]_t^n = \sum_{i=1}^{m_n} (B_{t_i} - B_{t_{i-1}})^2 \to t \quad \text{in } L^2. \tag{2}$$

Show that if the subdivision is dyadic, then the convergence is also almost sure.

Problem 14*. This question continues with the ideas of Problem 13; we will show that the convergence in (2) is almost sure as the subdivisions are nested, for a single fixed t.

Suppose that, for each $n \ge 3$, Δ_n is obtained from Δ_{n-1} by adding a new point, let us say $t_i(n)$.

i. Show that there exists a Brownian motion B' and a random variable ν, with $\mathbb{P}(\nu = \pm 1) = \frac{1}{2}$, such that

$$B_s = B'_{\min(s,t_i(n))} + \nu(B'_s - B'_{\min(s,t_i(n))})$$

and such that ν is independent of B'.

ii. Show that, for $k \geq n$, $[B]_t^k = [B']_t^k$, and compute $[B]_t^n - [B]_t^{n-1}$ in terms of $[B']_t^n - [B']_t^{n-1}$ and ν.

iii. Write $\mathcal{G}_n$ for the σ-algebra $\mathcal{G}_n = \sigma([B]_t^m : m \geq n)$. Deduce from the steps above that

$$\mathbb{E}[[B]_t^{n-1} | \mathcal{G}_n] = [B]_t^n \quad \text{almost surely.}$$

Conclude that

$$[B]_t^n \to t \quad \text{almost surely.}$$

Problem 15*. (Law of the Iterated Logarithm) Let $(B_t)_{t\geq 0}$ be a standard Brownian motion starting at 0, and for $t \geq 0$, let

$$S_t = \sup_{s \leq t} B_s. \tag{3}$$

i. Fix $\epsilon > 0$, and consider $t_n = (1+\epsilon)^n$. Show that, almost surely,

$$S_{t_n} \leq (1+\epsilon)\sqrt{2t_n \log\log t_n} \qquad \text{for all } n \text{ large enough.} \tag{4}$$

Hence, show that

$$\limsup_{t\to\infty} \frac{S_t}{\sqrt{2t\log\log t}} \leq 1 \qquad \text{almost surely.} \tag{5}$$

ii. Let $\theta > 1$, $t_n = \theta^n$, and fix $0 < \alpha < \sqrt{1-\theta^{-1}}$. Show that, almost surely,

$$B_{t_n} - B_{t_{n-1}} \geq \alpha\sqrt{2t_n \log\log t_n} \qquad \text{infinitely often.} \tag{6}$$

Conclude that

$$\limsup_{t\to\infty} \frac{S_t}{\sqrt{2t\log\log t}} \geq 1 \qquad \text{almost surely.} \tag{7}$$

iii. Finally, deduce that

$$\limsup_{t\to 0} \frac{B_t}{\sqrt{2t\log\log \frac{1}{t}}} = 1 \qquad \text{almost surely.} \tag{8}$$