Stochastic Calculus and Applications (Lent 2018)

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Primary references:
J.-F. Le Gall, Brownian Motion, Martingales, and Stochastic Calculus, Springer
N. Berestycki, Stochastic Calculus and Applications, Lecture Notes
J. Miller, Stochastic Calculus and Applications, Lecture Notes
V. Silvestri, Stochastic Calculus and Applications, Lecture Notes
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1. Introduction

1.1 Motivation

ODE: $\dot{x}(t) = F(x(t))$ — fundamental in analysis

SDE: $\dot{x}(t) = F(x(t)) + \eta(t)$

random noise

What should η be?

- For $|t-s| \gg 0$, $\eta(t)$ and $\eta(s)$ should be essentially independent.
- Idealisation: $\eta(t)$ and $\eta(s)$ should be independent for $t \neq s$.

Such an η exists, White Noise, but it is only a generalised function (random Schwartz distribution).

Even if $F=0$, to make sense of

$$\dot{x} = \eta, \quad \text{i.e.} \quad x(t) - x(0) = \int_0^t \eta(s)\,ds, \qquad (*)$$

deterministically, η should be at least a signed measure.

- White Noise is not a random signed measure.
- If $(*)$ held, for any $0 = t_0 < t_1 < t_2 < \cdots$, the increments

$$x(t_i) - x(t_{i-1}) = \int_{t_{i-1}}^{t_i} \eta(s)\,ds$$

should be independent and their variance should be $\sim |t_i - t_{i-1}|$.

So x should be Brownian Motion.

In which sense can we make sense of this?

1.2. The Wiener Integral

Defn. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Then $S \subset L^2(\Omega, \mathcal{F}, \mathbb{P})$ is a Gaussian space if S is a closed linear subspace and any $X \in S$ is a centred Gaussian random variable.

Prop. Let H be any separable Hilbert space. Then there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a Gaussian space $S \subseteq L^2(\Omega, \mathcal{F}, \mathbb{P})$ and an isometry $I : H \to S$.

Proof. Let $(e_i)_{i=1}^{\infty}$ be a Hilbert basis for H and let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space on which a sequence of independent random variables X_i with $X_i \sim N(0,1)$ is defined.
For $f \in H$, set

$$I(f) = \sum_{i=1}^{\infty} (f, e_i) X_i \in L^2(\Omega, \mathcal{F}, \mathbb{P}).$$

Indeed, $\mathbb{E}\left(\sum_{i=1}^{k} (f,e_i) X_i - \sum_{i=1}^{\ell} (f,e_i) X_i \right)^2 \leq \sum_{i=k}^{\ell} |(f,e_i)|^2 \to 0$ since $f \in H$, so the sequence $\left(\sum_{i=1}^{k} (f,e_i) X_i \right)_k$ is Cauchy and the above limit exists in $L^2(\Omega, \mathcal{F}, \mathbb{P})$. [By the MCT, it also exists a.s.].
The map I is an isometry since it maps the orthonormal basis (e_i) to the orthonormal system (X_i) in $L^2(\Omega, \mathcal{F}, \mathbb{P})$.

That I is an isometry means that for every $f \in H$, there is a random variable $I(f) \sim N(0, (f,f)_H)$, $I(\alpha f + \beta g) = \alpha I(f) + \beta I(g)$ a.s.,

$$\mathbb{E}(I(f) I(g)) = (f,g)_H.$$

Defn. A Gaussian white noise on $\mathbb{R}_+$ is an isometry N from $L^2(\mathbb{R}_+)$ into a Gaussian space. For $A \subset \mathbb{R}_+$, write $N(A) = N(1_A)$.

Prop
- For $A \subset \mathbb{R}_+$ with $|A| < \infty$, $N(A)$ is $\mathcal{N}(0, |A|)$.
- If $A, B \subset \mathbb{R}_+$, $A \cap B = \emptyset$ then $N(A)$ and $N(B)$ are independent.
- If $A = \bigcup_{i=1}^{\infty} A_i$ for disjoint sets A_i with $|A_i| < \infty$ and $|A| < \infty$, then

$$N(A) = \sum_{i=1}^{\infty} N(A_i) \quad \text{in } L^2 \text{ and a.s.} \qquad (*)$$

Proof of $(*)$. $M_n = \sum_{i=1}^{n} N(A_i)$ is a martingale and bounded in L^2:

$$\mathbb{E} M_n^2 \underset{\uparrow}{=} \sum_{i=1}^{n} \mathbb{E} N(A_i)^2 = \sum_{i=1}^{n} |A_i| \leq |A|$$

disjointness: $\mathbb{E} N(A_i) N(A_j) = 0$ for $i \neq j$

Thus $\sum_{i=1}^{\infty} N(A_i)$ converges a.s. and in L^2. Similary, $\mathbb{E}\left(N(A) - \sum_{i=1}^{n} N(A_i)\right)^2 \to 0$, so $(*)$ holds.

Thus N looks like a random measure, but it is not!
Note that $(*)$ makes use of cancelations in the probability space (the exceptional set depends on $A, (A_i)$).

Define $B_t = N([0,t])$, for $t \geq 0$.

Fact. For any $t_1, \dots, t_n$, the vector (B_{t_i}) is jointly Gaussian and
$$\mathbb{E}(B_s B_t) = s \wedge t \quad \text{for all } s, t \geq 0.$$
Moreover, $B_0 = 0$ a.s. and $B_t - B_s$ is independent of $\sigma(B_r, r \leq s)$
$$B_t - B_s \sim \mathcal{N}(0, t-s) \quad \text{for } t \geq s. \qquad (\dagger)$$

Let $f \in L^2(\mathbb{R}_+)$ be a step function: $f = \sum_{i=1}^{n} f_i \mathbb{1}_{[s_i, t_i]}$, $t_i < s_{i+1}$.
Then

$$N(f) = \sum_{i=1}^{n} f_i (B_{t_i} - B_{s_i}).$$

This motivates the notation

$$N(f) = \int f(s)\, dB_s.$$

If (B) was of finite variation for each $\omega \in \Omega$, the last inequality would make sense in the sense of the Lebesgue-Stieltjes integral. But it is not.

Defn. B is a standard Brownian motion if $(\dagger)$ holds and if (B_t) is continuous in t for every $\omega \in \Omega$.

1.3. The Lebesgue-Stieltjes Integral

For an interval $[0,T]$, we always use the Borel σ-algebra $\mathcal{B}([0,T])$ unless otherwise stated.

Defn. Let $T>0$.

- A signed measure μ on $[0,T]$ is the difference of two finite positive measures $\mu_\pm$ on $[0,T]$ with disjoint support. The decomposition $\mu = \mu_+ - \mu_-$ is called the Hahn-Jordan decomposition of μ.
- The total variation of a signed measure $\mu = \mu_+ - \mu_-$ is the positive measure $|\mu| = \mu_+ + \mu_-$.

Prop. (Hahn-Jordan). For any finite positive measures μ_1, μ_2 on $[0,T]$ there is a signed measure μ s.t. $\mu = \mu_1 - \mu_2$.

Proof. Let $\nu = \mu_1 + \mu_2$. By the Radon-Nikodym Theorem, there are Borel functions $f_i \geq 0$ on $[0,T]$ s.t.

$$\mu_i(dt) = f_i(t)\,\nu(dt).$$

Let $f(t) = f_1(t) - f_2(t)$. Then

$$(\mu_1 - \mu_2)(dt) = f(t)\,\nu(dt) = f(t)^+\,\nu(dt) - f(t)^-\,\nu(dt)$$

where $f(t)^+ = f(t) \vee 0$, $f(t)^- = -f(t) \wedge 0$ are the positive and negative parts of $f(t)$. This gives the decomposition into disjoint measures.

Defn Let $T>0$.

- A function $a:[0,T]\to\mathbb{R}$ is <u>càdlàg</u>, or $a\in D([0,T])$, if $a(t_+)=a(t)$ for all t and $a(t_-)$ exists for all t. Here $a(t_\pm)=\lim_{s\to 0\pm} a(t+s)$.

- The <u>total variation</u> of a function $a:[0,T]\to\mathbb{R}$ is
$$V_a(t)=\sup\left\{\sum_{j=1}^{n}|a(t_j)-a(t_{j-1})| : n\in\mathbb{N},\ 0\le t_0<t_1<\dots<t_n=t\right\}+|a(0)|.$$

- A function $a:[0,T]\to\mathbb{R}$ is of <u>bounded variation</u>, or $a\in BV([0,T])$, if $V_a(T)<\infty$.

Prop.

(i) Let μ be a signed measure on $[0,T]$. Then $a(t)=\mu([0,t])$ is càdlàg and $V_a(t)=|\mu|[0,t]$. In particular, $a\in BV$.

(ii) Let $a:[0,T]\to\mathbb{R}$ be a càdlàg function of bounded variation. Then there is a signed measure μ s.t. $a(t)=\mu[0,t]$. In particular, $a\in D$.

Proof. We will use the fact that $f(t)=\nu[0,t]$ induces a bijection between increasing right-continuous functions on $[0,T]$ s.t. $f(0)\ge 0$ and finite positive measures on $[0,T]$. In particular such f are càdlàg.

(i) Let $\mu=\mu_+-\mu_-$ be the Jordan decomposition of μ. Then
$$a(t)=\underbrace{\mu_+[0,t]}_{a_+(t)}-\underbrace{\mu_-[0,t]}_{a_-(t)} \quad \text{is càdlàg}$$
since $a_\pm$ are increasing right-continuous functions.

Claim: $v_a(t) \le |\mu|[0,t]$

For any subdivision $0=t_0<t_1<\cdots<t_n=t$,

$$|a(0)| + \sum_{i=1}^{n} |a(t_i)-a(t_{i-1})| = \mu(\{0\}) + \sum_{i=1}^{n} |\mu(t_{i-1},t_i]| \le |\mu|[0,t]$$

$\Rightarrow$ $v_a(t) \le |\mu|[0,t]$

Claim: For any nested sequence of subdivisions $0=t_0^{(m)}<t_1^{(m)}<\cdots<t_{n_m}^{(m)}=t$, with $\max_i |t_i^{(m)}-t_{i-1}^{(m)}| \to 0$,

$$|\mu|[0,t] = |a(0)| + \lim_{m\to\infty} \sum_{i=1}^{n_m} |a(t_i^{(m)})-a(t_{i-1}^{(m)})|$$

In particular, $v_a(t) \ge |\mu|[0,t]$.

Consider the probability measure $P(ds) = \dfrac{|\mu|(ds)}{|\mu|(0,t]}$ on $(0,t]$

Let $\mathcal{F}_m = \sigma\left((t_{i-1}^{(m)}, t_i^{(m)}],\ 1\le i\le n_m\right)$. Note $\mathcal{F}_{m+1} \supset \mathcal{F}_m$.

Let $X = \dfrac{d\mu}{d|\mu|} = 1_{\operatorname{supp}\mu_+} - 1_{\operatorname{supp}\mu_-}$. Let $X_m = \mathbb{E}(X\,|\,\mathcal{F}_m)$. Then

$$X_m(s) = \frac{\mu(t_{i-1}^{(m)}, t_i^{(m)}]}{|\mu|(t_{i-1}^{(m)}, t_i^{(m)}]} = \frac{a(t_i^{(m)})-a(t_{i-1}^{(m)})}{|\mu|(t_{i-1}^{(m)}, t_i^{(m)}]} \quad \text{for } s\in (t_{i-1}^{(m)}, t_i^{(m)}].$$

Since (X_m) is a bounded martingale, also $X_m \to Y$ (some Y) in L^1 and a.s.
Since $\bigvee \mathcal{F}_m = \mathcal{B}(0,t]$, it follows that $X=Y$ a.s.

$\Rightarrow$ $\mathbb{E}|X_m| \longrightarrow \mathbb{E}|X| = 1$

$$\Leftrightarrow \frac{1}{\mu(0,t]} \sum_{i=1}^{n_m} |a(t_i^{(m)})-a(t_{i-1}^{(m)})| \to 1.$$

This is the claim.

(ii) Let a be as in (ii). WLOG $a(0)=0$. Define

$$a_\pm(t) = \tfrac{1}{2}(V_a(t) \pm a(t)).$$

Claim: $a_\pm$ are increasing.

For any subdivision $0=t_0<t_1<\cdots<t_n=t$ of $[0,t]$, and $s>t$,

$$2a_\pm(s) = V_a(s) \pm a(s) \geq \underbrace{|a(0)| + \sum_{i=1}^{n}|a(t_i)-a(t_{i-1})|}_{\geq V_a(t)-\varepsilon} + \underbrace{|a(s)-a(t)| \pm a(s)}_{\geq \pm a(t)}$$

for sufficiently fine subdivision

$$\geq 2a_\pm(t) - \varepsilon$$

$\Rightarrow a_\pm(s) \geq a_\pm(t)$, i.e., $a_\pm$ are increasing.

Example Sheet: V_a is right-continuous

$\Rightarrow a_\pm$ is right-continuous

$\Rightarrow a_\pm(t) = \tilde{\mu}_\pm[0,t]$ for finite positive measures $\tilde{\mu}_\pm$.

Let $\mu = \tilde{\mu}_+ - \tilde{\mu}_-$. Then μ is a signed measure,

$$a(t) = a_+(t) - a_-(t) = \mu[0,t].$$

Example. Let $a : [0,1] \to \mathbb{R}$ be given by

$$a(t) = \begin{cases} 1 & (t < \frac{1}{2}) \\ 0 & (t \geq \frac{1}{2}). \end{cases}$$

Then $V_a(1) = 2$. The associated signed measure is

$$\mu = \delta_0 - \delta_{\frac{1}{2}}, \qquad |\mu| = \delta_0 + \delta_{\frac{1}{2}}$$

Note that $a(0) \neq 0$ is interpreted as a jump at 0, i.e., of the extension of a to $t < 0$ by $a(t) = 0$. In practice, we will be interested in measures without atom at 0. i.e., $a(0) = 0$.

Defn. Let $a : [0,T] \to \mathbb{R}$ be càdlàg of bounded variation, and let μ be the associated signed measure. For $h \in L^1([0,T], |\mu|)$, the Lebesgue-Stieltjes Integral is defined by

$$\int_s^t h(s)\, da(s) = \int_{(s,t]} h(s)\, \mu(ds), \qquad 0 \leq s < t \leq T$$

$$\int_s^t h(s)\, |da(s)| = \int_{(s,t]} h(s)\, |\mu|(ds)$$

We write

$$(h \cdot a)(t) = \int_0^t h(s)\, da(s).$$

Defn. A càdlàg function $a : [0,\infty) \to \mathbb{R}$ is of finite variation if $a|_{[0,T]} \in BV[0,T]$ for every $T > 0$.

Fact. Let $a:[0,T]\to\mathbb{R}$ be càdlàg and BV, $h\in L^1([0,T],|da|)$. Then

$$\left|\int_0^t h(s)\,da(s)\right| \le \int_0^t |h(s)|\,|da(s)|$$

and the function $h\cdot a:[0,T]\to\mathbb{R}$ defined by

$$(h\cdot a)(t) = \int_0^t h(s)\,da(s)$$

is càdlàg and BV with signed measure $h(s)\,da(s)$, $|h(s)\,da(s)| = |h(s)|\,|da(s)|$.

Prop. Let a be of finite variation and h left-continuous, bounded. Then

$$\int_0^t h(s)\,da(s) = \lim_{n\to\infty} \sum_{i=1}^{n_m} h(t_{i-1}^{(m)})\left(a(t_i^{(m)}) - a(t_{i-1}^{(m)})\right)$$

$$\int_0^t h(s)\,|da(s)| = \lim_{n\to\infty} \sum_{i=1}^{n_m} h(t_{i-1}^{(m)})\,\left|a(t_i^{(m)}) - a(t_{i-1}^{(m)})\right|$$

for any sequence of subdivisions $0=t_0^{(m)}<t_1^{(m)}<\dots<t_{n_m}^{(m)}=t$ with step size $\max_{i\le n_m}|t_i^{(m)}-t_{i-1}^{(m)}|$ tending to 0.

Proof. Let $h_m(0)=0$,

$$h_m(s) = h(t_{i-1}^{(m)}) \quad \text{if } s\in(t_{i-1}^{(m)}, t_i^{(m)}].$$

$\Rightarrow$ $h(s) = \lim_{m\to\infty} h_m(s)$ by left-continuity

$$\Rightarrow \lim_{n\to\infty} \sum_{i=1}^{n_m} h(t_{i-1}^{(m)})\left(a(t_i^{(m)})-a(t_{i-1}^{(m)})\right) = \int_{(0,t]} h_m(s)\,\mu(ds) \to \int h\,\mu(ds)$$

by the DCT. The statement about $|da(s)|$ is left as an exercise (use claim from p.7 for nested subdivisions).

2. Semimartingales

From now on, $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ is a filtered probability space.

Defn. A càdlàg adapted process X is a map $X: \Omega \times [0,\infty) \to \mathbb{R}$ s.t.

(i) X is càdlàg, i.e., $X(\omega, \cdot): [0,\infty) \to \mathbb{R}$ is càdlàg for all $\omega \in \Omega$

(ii) X is adapted, i.e., $X_t = X(\cdot, t)$ is $\mathcal{F}_t$-measurable for all $t \geq 0$.

Notation: write $X \in \mathcal{F}$ to denote that a random variable X is measurable w.r.t. a σ-algebra $\mathcal{F}$

2.1. Finite variation processes

Defn. (i) A càdlàg adapted process A is a finite variation process if $A(\omega, \cdot): [0,\infty) \to \mathbb{R}$ has finite variation for all $\omega \in \Omega$.

(ii) The total variation process V of a finite variation process A is defined by

$$V_t = \int_0^t |dA_s| .$$

Fact. The total variation process V of a càdlàg adapted process A is càdlàg adapted and it is increasing.

Proof. We only need to check that V is adapted. (The other properties follow from Section 1.3.)

Let $0 = t_0^{(m)} < t_1^{(m)} < \cdots < t_{n_m}^{(m)} = t$ be a nested sequence of subdivisions of $[0,t]$ with $\lim_{m\to\infty} \max_i |t_i^{(m)} - t_{i-1}^{(m)}| = 0$. We have seen

$$V_t = \lim_{m\to\infty} \sum_{i=1}^{n_m} \underbrace{|A_{t_i^{(m)}} - A_{t_{i-1}^{(m)}}|}_{\in \mathcal{F}_t} + |A(0)| \in \mathcal{F}_t$$

Thus V is adapted.

Defn. Let A be a finite variation process and H be a process s.t.

$$\forall \omega \in \Omega \ \forall t \geq 0 : \int_0^t |H_s(\omega)| \, |dA_s(\omega)| < \infty .$$

Then define a process $((H \cdot A)_t)_{t \geq 0}$ by

$$(H \cdot A)_t = \int_0^t H_s \, dA_s .$$

For the process $H \cdot A$ to be adapted, we need a condition on H.

Defn. A process $H : \Omega \times [0,\infty) \to \mathbb{R}$ is previsible if it is measurable w.r.t. the previsible σ-algebra generated by the sets

$$E \times (s,t], \qquad E \in \mathcal{F}_s, \ s < t.$$

Defn. A process $H : \Omega \times [0,\infty) \to \mathbb{R}$ is simple, $H \in \mathcal{E}$, if

$$H(\omega, t) = \sum_{i=1}^{n} H_{i-1}(\omega) \, 1_{(t_{i-1}, t_i]}(t)$$

for bounded random variables $H_{i-1} \in \mathcal{F}_{i-1}$ and $0 = t_0 < t_1 < \cdots < t_n$.

Fact. Simple processes and their pointwise limits are previsible.

Fact. Let X be a càdlàg adapted process. Then $H_t = X_{t-}$ defines a left-continuous process which is previsible.

Proof. Since X is càdlàg adapted, it is clear that H is left-continuous and adapted.
Since H is left-continuous, it can be approximated by simple processes. Let

$$H_t^n = \sum_{i=1}^{n2^n} H_{(i-1)2^{-n}} 1_{((i-1)2^{-n}, i2^{-n}]}(t) \wedge n.$$

Then $H_t^n \to H_t$ for all t by left-continuity and that H is previsible follows from the previous fact.

Exercise. Let H be previsible. Then $H_t \in \mathcal{F}_{t-}$ where $\mathcal{F}_{t-} = \sigma(\mathcal{F}_s : s < t)$.

Example.

- Brownian motion is previsible since continuous.
- A Poisson process (N_t) is not previsible since $N_t \notin \mathcal{F}_{t-}$.

Prop. Let A be a finite variation process and H previsible. Then $H\cdot A$ is a finite variation process.

Proof. By Section 1.3, for every $\omega\in\Omega$, $(H\cdot A)(\omega,\cdot)$ is of finite variation and thus also càdlàg.

Claim: $H\cdot A$ is adapted

First, $H\cdot A$ is adapted if $H(\omega,s)=1_{(u,v]}1_E(\omega)$ for $u<v$, $E\in\mathcal{F}_u$:

$$(H\cdot A)(\omega,t)=1_E(\omega)\big(A(\omega,t\wedge v)-A(\omega,t\wedge u)\big)\Rightarrow (H\cdot A)_t\in\mathcal{F}_t.$$

Thus $H\cdot A$ is adapted for $H=1_F$, when $F\in\Pi$,

$$\Pi=\{E\times(u,v]:\ E\in\mathcal{F}_u,\ u<v\}\subset\Omega\times[0,\infty).$$

Clearly, Π is a π-system (closed under intersection, nonempty), generating the previsible σ-algebra $\mathcal{P}$. Let

$$\mathcal{D}=\{H:\Omega\times[0,\infty)\to\mathbb{R}:\ H\cdot A \text{ is adapted}\}.$$

Then: $1\in\mathcal{D}$, $1_H\in\mathcal{D}$ for $H\in\Pi$ by the above, and if $0\le H_n\in\mathcal{D}$ with $H_n\uparrow H$ then $H\in\mathcal{D}$ since measurability is closed under limits.

Thus $\mathcal{D}$ is a monotone class. By the monotone class theorem, $\mathcal{D}$ contains all bounded previsible processes.

2.2. Local martingales

From now on, we assume that $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ satisfies the usual conditions (recall motivation from Advanced Probability):

- $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets,
- $(\mathcal{F}_t)$ is right-continuous, i.e., $\mathcal{F}_t = \mathcal{F}_{t+} = \bigcap_{s>t} \mathcal{F}_s$ for all $t \geq 0$.

Thm (Optional Stopping Theorem) Let X be a càdlàg adapted integrable process. Then the following are equivalent:

(i) X is a martingale, i.e. $X_t \in L^1$ for every t and

$$\mathbb{E}(X_t | X_s) = X_s \quad \text{for all } t \geq s.$$

(ii) $X^T = (X^T_t) = (X_{T \wedge t})$ is a martingale for all stopping times T, (where T is a stopping time if $\{T \leq t\} \in \mathcal{F}_t$ for all t).

(iii) For all stopping times T, S with T bounded, $X_T \in L^1$ and

$$\mathbb{E}(X_T | \mathcal{F}_S) = X_{S \wedge T} \text{ a.s.}$$

(iv) For all bounded stopping times T, $X_T \in L^1$ and $\mathbb{E}(X_T) = \mathbb{E}(X_0)$.

For X uniformly integrable, (iii) & (iv) hold for all stopping times.

Defn. A càdlàg adapted process X is a local martingale if there exists a sequence of stopping times (T_n) s.t. $T_n \uparrow \infty$ as $n \to \infty$ and X^{T_n} is martingale for every n. The sequence (T_n) is said to reduce X.

Example. (i) Every martingale is a local martingale. (Take $T_n = n$ and use the OST.)

(ii) Let (B_t) be a standard Brownian motion on $\mathbb{R}^3$. Then $(X_t)_{t\geq 1} = (1/|B_t|)_{t\geq 1}$ is a local martingale, but not a martingale.

Proof. It is true that ($\to$ Advanced Probability)

$$\sup_{t\geq 1} \mathbb{E} X_t^2 < \infty, \quad \mathbb{E} X_t \to 0, \quad M \text{ is a supermartingale.}$$

Since $\mathbb{E} X_t \to 0$, X cannot be a martingale.

To show that X is a local martingale, recall that for $f \in C_b^2$,

$$f(B_t) - f(B_1) - \frac{1}{2}\int_1^t \Delta f(B_s)\, ds =: M^f$$

is a martingale. Moreover, $\Delta \frac{1}{|x|} = 0$ for $x \neq 0$.

Let $T_n = \inf\{t \geq 1 : |B_t| < \frac{1}{n}\}$ and $f_n \in C_b^2$ with $f_n(x) = \frac{1}{x}$ for $x \geq \frac{1}{n}$.

Then

$$X_t^{T_n} - X_1^{T_n} = M^{f_n}_{t\wedge T_n}$$

so X^{T_n} is a martingale. To show that X is a local martingale, it remains to show that $T_n \to \infty$ a.s.

Let $S_m = \inf\{t \geq 1 : |B_t| > m\}$. Then, by OST ($X^{T_n}$ is a bounded martingale)

$$\mathbb{E}(X_{T_n \wedge S_m}) = \mathbb{E}(X_1) < \infty.$$

But also

$$\mathbb{E}(X_{T_n\wedge S_m}) = n\, \mathbb{P}(T_n < S_m) + \frac{1}{m} \overbrace{\mathbb{P}(T_n \geq S_m)}^{1-\mathbb{P}(T_n < S_m)} = \left(n - \frac{1}{m}\right) \mathbb{P}(T_n < S_m) + \frac{1}{m}.$$

$$\Rightarrow \mathbb{P}(T_n < S_m) = \frac{\mathbb{E}(X_1) - 1/m}{n - 1/m} \to \frac{\mathbb{E}(X_1)}{n} \text{ as } m \to \infty$$

$$\Rightarrow \mathbb{P}(T_n < \infty) \leq \frac{\mathbb{E}(X_1)}{n}$$

$$\Rightarrow \mathbb{P}\left(\lim_{n\to\infty} T_n < \infty\right) = 0.$$

The next proposition shows that X is also a supermartingale.

<u>Prop.</u> Let X be a local martingale and $X_t \geq 0$ for all t. Then X is a supermartingale.

<u>Proof.</u> Let (T_n) be a reducing sequence. Then

$$\mathbb{E}(X_t \mid \mathcal{F}_s) = \mathbb{E}\left(\liminf_{n\to\infty} X_{t\wedge T_n} \mid \mathcal{F}_s\right)$$

$$\leq \liminf_{n\to\infty} \mathbb{E}(X_{t\wedge T_n} \mid \mathcal{F}_s) \text{ by conditional Fatou}$$

$$= \liminf_{n\to\infty} X_{s\wedge T_n} = X_s.$$

<u>Lemma</u> (Example Sheet 1). Let $X \in L^1(\Omega, \mathcal{F}, \mathbb{P})$. Then the set

$$\mathcal{X} = \{\mathbb{E}(X \mid \mathcal{G}) : \mathcal{G} \subset \mathcal{F} \text{ is a sub-}\sigma\text{-algebra}\}$$

is uniformly integrable (UI), i.e.,

$$\sup_{Y\in\mathcal{X}} \mathbb{E}(|Y| \mathbb{1}_{|Y|>\lambda}) \to 0 \text{ as } \lambda \to \infty.$$

<u>Recall</u> (Vitali). $X_n \to X$ in L^1 iff (X_n) is UI and $X_n \to X$ in prob.

Prop. The following are equivalent:

(a) X is a martingale

(b) X is a local martingale and for all $t \geq 0$, the set

$$\mathcal{X}_t = \{X_T : T \text{ is a stopping time with } T \leq t\} \text{ is UI.}$$

Proof. (a) $\Rightarrow$ (b). Let X be a martingale. Then by OST

$$X_T = \mathbb{E}(X_t \mid \mathcal{F}_T), \text{ for any bounded stopping time } T \leq t.$$

By the previous lemma, $\mathcal{X}_t$ is UI.

(b) $\Rightarrow$ (a). Let X be a local martingale with reducing sequence (T_n) and assume that $\mathcal{X}_t$ is UI for any $t \geq 0$.

To prove that X is a martingale, it suffices to prove that $\mathbb{E}(X_T) = \mathbb{E}(X_0)$ for any bounded stopping time T.

Let T be a bounded stopping time with $T \leq t$. Then

$$\mathbb{E}(X_0) = \mathbb{E}(X_0^{T_n}) = \mathbb{E}(X_T^{T_n}) = \mathbb{E}(X_{T \wedge T_n}).$$

By assumption, $\{X_{T \wedge T_n} : n \geq 0\}$ is UI. Since $T \wedge T_n \to T$ a.s., therefore $X_{T \wedge T_n} \to X_T$ in L^1.

$\Rightarrow \mathbb{E}(X_0) = \mathbb{E}(X_T)$.

Thus X is a martingale.

Cor. (i) Every bounded local martingale X is a martingale.
(ii) If there is $Z \in L^1$ s.t. $|X_t| \le Z$ for all t, then X is a martingale.

Prop. Let X be a continuous local martingale with $X_0 = 0$.
Let $S_n = \inf\{t \ge 0 : |X_t| = n\}$. Then S_n is a stopping time, $S_n \uparrow \infty$, and X^{S_n} is a (bounded) martingale. In particular, (S_n) reduces X.

Proof. Claim: S_n is a stopping time

$$\{S_n \le t\} = \bigcap_{k \in \mathbb{N}} \left\{\sup_{s \le t} |X_s| > n - \frac{1}{k}\right\} \underset{\text{continuity}}{=} \bigcap_{k \in \mathbb{N}} \bigcup_{\substack{s \le t \\ s \in \mathbb{Q}}} \left\{|X_s| > n - \frac{1}{k}\right\} \in \mathcal{F}_t$$

Claim: $S_n \uparrow \infty$.

$$\underbrace{\sup_{s \le t} |X_s|}_{< \infty \text{ for every } (\omega, t) \text{ by continuity}} \le n \Rightarrow S_n \ge t$$

Claim: X^{S_n} is a martingale
By OST, $X^{T_k \wedge S_n}$ is a martingale (when (T_k) reduces X).
$\Rightarrow X^{S_n}$ is a local martingale
$|X^{S_n}| \le n \Rightarrow X$ is a bounded local martingale $\Rightarrow X$ is a martingale.

Thm. Let X be a continuous local martingale, $X_0=0$. If X is also a finite variation process then $X_t=0$ $\forall t$ a.s.

Proof. Let (V_t) be the total variation process of X and

$$S_n = \inf\{t\ge 0 : \underbrace{\int_0^t |dX_s|}_{V_t} \ge n\}.$$

Then S_n is a stopping time and $S_n \uparrow \infty$ as $n\to\infty$.

Moreover, X^{S_n} is a local martingale by OST, and X^{S_n} is bounded:

$$|X_t^{S_n}| < \int_0^{t\wedge S^n} |dX_s| < n.$$

$\Rightarrow X^{S_n}$ is a martingale.

Let $0=t_0<\cdots<t_k=t$ be a subdivision of $[0,t]$. Then

$$\mathbb{E}(X_t^{S_n})^2 = \sum_{i=1}^{k} \mathbb{E}\left((X_{t_i}^{S_n} - X_{t_{i-1}}^{S_n})^2\right) \text{ since } X^{S_n} \text{ is a martingale}$$

$$\le \mathbb{E}\Big(\underbrace{\max_{1\le i\le k} |X_{t_i}^{S_n} - X_{t_{i-1}}^{S_n}|}_{\text{bounded}} \underbrace{\sum_{i=1}^{k} |X_{t_i}^{S_n} - X_{t_{i-1}}^{S_n}|}_{\le \int_0^{t\wedge S_n} |dX_s| \le n}\Big)$$

Take $\max_{i\le k} |t_i - t_{i-1}| \to 0$. By continuity and DCT,

$$\mathbb{E}(X_t^{S_n})^2 = 0$$

$\Rightarrow X_{t\wedge S_n} = 0$ a.s. $\Rightarrow X_t = 0$ a.s.

X continuous $\Rightarrow X=0$ $\forall t$ a.s.

2.3. Square integrable martingales

<u>Defn</u>. Let

$$\mathcal{M}^2 = \{X : [0,\infty) \times \Omega \to \mathbb{R} : X \text{ is a càdlàg martingale}, \sup_{t \geq 0} \mathbb{E} X_t^2 < \infty\} / \sim$$

$$\mathcal{M}_c^2 = \{X \in \mathcal{M}^2 : X(\cdot, \omega) \text{ is continuous for every } \omega \in \Omega\} / \sim$$

where indistinguishable processes are identified, and set

$$\|X\|_{\mathcal{M}^2} = (\mathbb{E}(X_\infty^2))^{1/2}.$$

Here recall that for $X \in \mathcal{M}^2$, the martingale convergence theorem implies that

$$X_t \longrightarrow X_\infty \quad \text{a.s. and in } L^2$$

Moreover, $(X_t^2)_{t \geq 0}$ is a submartingale, so $t \mapsto \mathbb{E} X_t^2$ is increasing, and

$$\mathbb{E} X_\infty^2 = \sup_{t \geq 0} \mathbb{E} X_t^2.$$

Doob's L^2 inequality implies

$$\mathbb{E}\left(\sup_{t \geq 0} X_t^2\right) \leq 4 \mathbb{E} X_\infty^2.$$

In particular, $\|X\|_{\mathcal{M}^2} = 0$ implies that $X = 0$. This makes $\|\cdot\|_{\mathcal{M}^2}$ a norm (the other properties are clear).

This norm comes from the inner product $\mathbb{E}(X_\infty Y_\infty)$ on $\mathcal{M}^2$.

Thm. $\mathcal{M}^2$ is a Hilbert space and $\mathcal{M}^2_c$ is a closed subspace.

Proof. We need to show that $\mathcal{M}^2$ is complete. Thus let $(X^n) \subset \mathcal{M}^2$ be a Cauchy sequence:

$$\mathbb{E}(X^n_\infty - X^m_\infty)^2 \to 0 \text{ as } n,m \to \infty.$$

By passing to a subsequence, we may assume that

$$\mathbb{E}(X^n_\infty - X^{n-1}_\infty)^2 \leq 2^{-n}$$

and it suffices to prove that the subsequence converges to show that the original sequence converges.

$$\mathbb{E}\Big(\sum_n \sup_{t\geq 0} |X^n_t - X^{n-1}_t|\Big) \overset{CS}{\leq} \sum_n \mathbb{E}\Big(\sup_{t\geq 0} |X^n_t - X^{n-1}_t|^2\Big)^{1/2}$$

$$\overset{\text{Doob}}{\leq} \sum_n 2\mathbb{E}(|X^n_\infty - X^{n-1}_\infty|^2)^{1/2} \leq \sum_n 2^{1-\frac{n}{2}} < \infty$$

$$\Rightarrow \sum_{n=1}^\infty \sup_{t\geq 0} |X^n_t - X^{n-1}_t| < \infty \text{ a.s.}$$

$\Rightarrow (X^n)$ is Cauchy in $D[0,\infty), \|\cdot\|_\infty$ a.s.

$\Rightarrow \|X^n - X\|_\infty \to 0$ a.s. for some $X \in D[0,\infty)$

Set $X=0$ outside the a.s. event. Then $X \in D[0,\infty)$ everywhere.

Claim: $\mathbb{E}\big(\sup_{t\geq 0} |X^n - X|^2\big) \to 0$

$$\mathbb{E}\Big(\sup_t |X^n - X|^2\Big) = \mathbb{E}\Big(\lim_{m\to\infty} \sup_t |X^n - X^m|^2\Big)$$

$$\overset{\text{Fatou}}{\leq} \liminf_{m\to\infty} \mathbb{E}\Big(\sup_t |X^n - X^m|^2\Big) \leq 4\mathbb{E}((X^n_\infty - X^m_\infty)^2) \to 0.$$

Claim: X is a martingale

$$\|\mathbb{E}(X_t|\mathcal{F}_s) - X_s\|_{L^2} \leq \|\mathbb{E}(X_t - X_t^n | \mathcal{F}_s)\|_{L^2} + \|X_s^n - X_s\|_{L^2}$$

$$= \|X_t - X_t^n\|_{L^2} + \|X_s^n - X_s\|_{L^2}$$

$$\leq 2\,\mathbb{E}\left(\sup_t |X_t - X_t^n|^2\right)^{1/2} \to 0$$

Thus $X \in \mathcal{M}^2$ and we have shown that $\mathcal{M}^2$ is complete.
Clearly, $\mathcal{M}_c^2$ is a subspace of $\mathcal{M}^2$ and completeness follows from the same argument replacing $D[0,\infty)$ by $C[0,\infty)$.

2.4. Quadratic variation

Defn. For a sequence of processes (X^n) and a process X,

$$X^n \to X \text{ u.c.p. (uniformly on compact sets in probability)}$$

iff

$$\mathbb{P}\Big(\sup_{s\in[0,t]} |X^n_s - X_s| > \varepsilon\Big) \xrightarrow{n\to\infty} 0 \quad \forall t>0 \ \forall \varepsilon>0.$$

Thm. Let M be a continuous local martingale. Then there exists a (unique up to indistinguishability) continuous adapted increasing process $(\langle M\rangle_t)_{t\geq 0}$ such that $\langle M\rangle_0 = 0$ and that $M_t^2 - \langle M\rangle_t$ is a continuous local martingale. Moreover,

$$\langle M\rangle_t = \lim_{n\to\infty} \langle M\rangle_t^{(n)}, \quad \text{where } \langle M\rangle_t^{(n)} = \sum_{i=1}^{\lceil 2^n t\rceil} (M_{i2^{-n}} - M_{(i-1)2^{-n}})^2,$$

with convergence $\langle M\rangle^{(n)} \to \langle M\rangle$ u.c.p.

Defn. $\langle M\rangle$ is the quadratic variation of M.

Proof. Assume $M_0=0$ (by replacing M_t by $M_t - M_0$).

Uniqueness: Suppose (A_t) and $(\hat{A}_t)$ obey the condition for $\langle M\rangle$. Then

$$\underbrace{A_t - \hat{A}_t}_{\text{finite variation}} = \underbrace{(M_t^2 - A_t) - (M_t^2 - \hat{A}_t)}_{\text{continuous martingale}}$$

$\Rightarrow A - \hat{A} = 0$ a.s.

Existence for M bounded. Assume $M(\omega,t) \le C$ for all (ω,t).

Then $M \in \mathcal{M}_c^2$. Fix $T>0$ deterministic. Let

$$X_t^n = \sum_{i=1}^{\lceil 2^n T \rceil} M_{(i-1)2^{-n}} \left(M_{i2^{-n}\wedge t} - M_{(i-1)2^{-n}\wedge t} \right).$$

So that

$$\langle M \rangle_{k2^{-n}}^{(n)} = \sum_{i=1}^{k} \underbrace{\left(M_{i2^{-n}} - M_{(i-1)2^{-n}} \right)^2}$$

$$\underbrace{M_{i2^{-n}}} \left(M_{i2^{-n}} - M_{(i-1)2^{-n}} \right) - M_{(i-1)2^{-n}} \left(M_{i2^{-n}} - M_{(i-1)2^{-n}} \right)$$

$$- M_{(i-1)2^{-n}} + \left(M_{i2^{-n}} + M_{(i-1)2^{-n}} \right)$$

$$= \sum_{i=1}^{k} \left(M_{i2^{-n}}^2 - M_{(i-1)2^{-n}}^2 \right) - 2X_{k2^{-n}}^n$$

$$= M_{k2^{-n}}^2 - 2X_{k2^{-n}}^n \qquad (*)$$

Note that, for every n, X^n is a continuous martingale, $X^n \in \mathcal{M}_c^2$.

Claim: (X^n) is Cauchy in $\mathcal{M}_c^2$

For $n \ge m$,

$$X_\infty^n - X_\infty^m = \sum_{i=1}^{\lceil 2^n T \rceil} \left(M_{(i-1)2^{-n}} - M_{\lfloor (i-1)2^{m-n} \rfloor 2^{-m}} \right) \left(M_{i2^{-n}} - M_{(i-1)2^{-n}} \right)$$

$$\Rightarrow \mathbb{E}\left((X_\infty^n - X_\infty^m)^2 \right) \underset{\text{orthogonal increments}}{=} \mathbb{E}\left(\sum_{i=1}^{\lceil 2^n T \rceil} \left(M_{(i-1)2^{-n}} - M_{\lfloor (i-1)2^{m-n} \rfloor 2^{-m}} \right)^2 \left(M_{i2^{-n}} - M_{(i-1)2^{-n}} \right)^2 \right)$$

$$\le \mathbb{E}\left(\sup_{|s-t| \le 2^{-m}} |M_t - M_s|^2 \underbrace{\sum_{i=1}^{\lceil 2^n T \rceil} \left(M_{i2^{-n}} - M_{(i-1)2^{-n}} \right)^2}_{\langle M \rangle_T^{(n)}} \right)$$

$$\Rightarrow \mathbb{E}(X_\infty^n - X_\infty^m)^2 \leq \mathbb{E}\Big(\sup_{|s-t|\leq 2^{-m}} |M_t - M_s|^4\Big)^{1/2} \mathbb{E}\Big(\big(\langle M\rangle_T^{(n)}\big)^2\Big)^{1/2}$$

Claim: $\mathbb{E}\Big(\sup_{|s-t|\leq 2^{-m}} |M_t - M_s|^4\Big) \to 0$ as $m\to\infty$.

Indeed, $|M_t - M_s|^4 \leq 16C^4$,

$\sup_{|s-t|\leq 2^{-m}} |M_t - M_s| \to 0$ as $m \to \infty$ by uniform continuity

$$\Rightarrow \mathbb{E}\Big(\sup_{|s-t|\leq 2^{-m}} |M_t - M_s|^4\Big) \to 0 \text{ by DCT}$$

Claim: $\mathbb{E}\Big(\big(\langle M\rangle_T^{(n)}\big)^2\Big) \leq 48C^4$

$$\mathbb{E}\Bigg(\Big(\sum_{i=1}^N (M_{i2^{-n}} - M_{(i-1)2^{-n}})^2\Big)^2\Bigg) = \sum_{i=1}^N \mathbb{E}\big((M_{i2^{-n}} - M_{(i-1)2^{-n}})^4\big)$$

$$+ 2\sum_{i=1}^N \underbrace{\mathbb{E}\Big((M_{i2^{-n}} - M_{(i-1)2^{-n}})^2 \sum_{k=i+1}^N (M_{k2^{-n}} - M_{(k-1)2^{-n}})^2\Big)}_{\mathbb{E}\big((M_{i2^{-n}} - M_{(i-1)2^{-n}})^2 (M_{N2^{-n}} - M_{i2^{-n}})^2\big)}$$

$$\leq 12C^2\, \mathbb{E}\Big(\sum_{i=1}^N (M_{i2^{-n}} - M_{(i-1)2^{-n}})^2\Big)$$

$$= 12C^2\, \mathbb{E}\big((M_{N2^{-n}} - M_0)^2\big)$$

$$\leq 48C^4$$

Upshot: $\|X^n - X^m\| = \mathbb{E}\big((X_\infty^n - X_\infty^m)^2\big)^{1/2} \to 0$, i.e. (X^n) is Cauchy.

$\Rightarrow X^n \xrightarrow{\mathcal{M}^2} X$ for some $X \in \mathcal{M}_c^2$.

Since $X^n \to X$ in $\mathcal{M}_c^2$, in particular $\| \sup_t |X_t^n - X_t| \|_{L^2} \to 0$.

Therefore $\sup_t |X_t^n - X_t| \to 0$ a.s. along a subsequence Λ.

Let $N \subset \Omega$ be the event on which the convergence fails, and set

$$A_t^{(T)} = \begin{cases} M_t^2 - 2X_t & \text{for } \omega \in \Omega \setminus N \\ 0 & \text{for } \omega \in N \end{cases}.$$

Then:

- $A^{(T)}$ is continuous, adapted since M and X are.
- $(M_{t\wedge T}^2 - A_{t\wedge T}^{(T)})_t$ is a martingale since X is.
- $A^{(T)}$ is increasing since $M_t^2 - 2X_t^n$ is increasing on $2^{-n}\mathbb{Z} \cap [0,T]$ and the convergence is uniform.

Claim: For any $T \geq 1$, $A_{t\wedge T}^{(T)} = A_{t\wedge T}^{(T+1)}$ for all t, a.s.

This follows from the same argument as the uniqueness.

Thus there is a process $(\langle M \rangle_t)_{t\geq 0}$ s.t. $\langle M \rangle_t = A_t^{(T)}$ for all $t \in [0,T]$ and all $T \in \mathbb{N}$ a.s.

Clearly, $\langle M \rangle$ is increasing and $M^2 - \langle M \rangle$ is a martingale.

This concludes the construction of $\langle M \rangle$, for M bounded.

Claim: $\langle M\rangle^{(n)} \to \langle M\rangle$ u.c.p.

Recall that $\langle M\rangle^{(n)}_t = M^2_{2^{-n}\lfloor 2^n t\rfloor} - 2X^n_{2^{-n}\lfloor 2^n t\rfloor}$

$\sup_{t\le T} |X^n_t - X_t| \to 0$ in L^2 and thus in prob.

$$\Rightarrow \sup_{t\le T} |\langle M\rangle_t - \langle M\rangle^{(n)}_t| \le \underbrace{\sup_{t\le T} |M^2_{2^{-n}\lfloor 2^n t\rfloor} - M^2_t|}_{\substack{\to 0 \text{ in prob.} \\ \text{by unif. cont.}}} + \underbrace{\sup_{t\le T} |X^n_{2^{-n}\lfloor 2^n t\rfloor} - X_{2^{-n}\lfloor 2^n t\rfloor}|}_{\substack{\to 0 \text{ in prob} \\ \text{by above}}} + \underbrace{\sup_{t\le T} |X_{2^{-n}\lfloor 2^n t\rfloor} - X_t|}_{\substack{\to 0 \text{ in prob.} \\ \text{by unif. cont.}}}$$

Existence for M a continuous local martingale. Let

$$T_n = \inf\{t\ge 0 : |M_t| \ge n\}.$$

Then (T_n) reduces M and M^{T_n} is a bounded martingale.

Let $A^n = \langle M^{T_n}\rangle$.

Then (A^n_t) and $(A^{n+1}_{t\wedge T_n})$ are indistinguishable by the uniqueness argument.

Thus there is a process $\langle M\rangle$ s.t. $\langle M\rangle_{t\wedge T_n}$ and A^n are indistinguishable for all n.

Clearly, $\langle M\rangle$ is increasing since the A^n are and $M^2_{t\wedge T_n} - \langle M\rangle_{t\wedge T_n}$ is martingale for every n. Thus $M^2 - \langle M\rangle$ is a local martingale.

Claim: $\langle M\rangle^{(n)} \to \langle M\rangle$ u.c.p.

We have seen that $\langle M^{T_k}\rangle^{(n)} \to \langle M^{T_k}\rangle$ u.c.p. for every k.

$$\Rightarrow \mathbb{P}\left(\sup_{t\le T} |\langle M\rangle^{(n)}_t - \langle M\rangle_t| > \varepsilon\right) \le \mathbb{P}(T_k < T) + \mathbb{P}\left(\sup_{t\le T} |\langle M^{T_k}\rangle^{(n)}_t - \langle M^{T_k}\rangle_t| > \varepsilon\right)$$

$\to 0$ as $n\to\infty$ since $T_k \to \infty$

This finishes the proof (finally).

Example. Let B be a standard Brownian motion. Then $B_t^2 - t$ is a martingale and thus $\langle B\rangle_t = t$.

Fact. Let M be a continuous local martingale and let T be a stopping time. Then a.s. for all $t \ge 0$

$$\langle M^T\rangle_t = \langle M\rangle_{t\wedge T}.$$

Proof. Since $M_t^2 - \langle M\rangle_t$ is a continuous local martingale, so is $M^2_{t\wedge T} - \langle M\rangle_{t\wedge T} = (M^T)^2_t - \langle M\rangle_{t\wedge T}$. Thus $\langle M\rangle_{t\wedge T} = \langle M^T\rangle_t$ by uniqueness.

Fact. Let M be a continuous local martingale with $M_0 = 0$. Then $M=0$ iff $\langle M\rangle = 0$.

Proof. If $\langle M\rangle = 0$, then M^2 is a continuous local martingale and positive, so a supermartingale. Thus $\mathbb{E} M_t^2 \le \mathbb{E} M_0^2 = 0$ for all t.

Prop. Let $M \in \mathcal{M}_c^2$. Then $M^2 - \langle M \rangle$ is a uniformly integrable martingale and $\|M - M_0\|_{\mathcal{M}^2} = \mathbb{E}(\langle M \rangle_\infty)^{1/2}$.

Proof. We will show that $\langle M \rangle_\infty \in L^1$. Then

$$|M_t^2 - \langle M \rangle_t| \leq \underbrace{\sup_{t \geq 0} M_t^2 + \langle M \rangle_\infty}_{\in L^1 \text{ (Doob's } L^2 \text{ inequality)}} \quad \text{for all } t \geq 0$$

Since $M^2 - \langle M \rangle$ is also a local martingale, this implies that $M^2 - \langle M \rangle$ is a UI martingale.

Claim: $\langle M \rangle_\infty \in L^1$.

Let $S_n = \inf\{t \geq 0 : \langle M \rangle_t \geq n\}$. Then $S_n \uparrow \infty$, S_n is a stopping time, and $\langle M \rangle_{t \wedge S_n} \leq n$.

$$\Rightarrow M_{t \wedge S_n}^2 - \langle M \rangle_{t \wedge S_n} \leq \underbrace{n + \sup_{t \geq 0} M_t^2}_{\in L^1}$$

$\Rightarrow M_{t \wedge S_n}^2 - \langle M \rangle_{t \wedge S_n}$ is a true martingale.

$$\Rightarrow \mathbb{E} M_{t \wedge S_n}^2 - \mathbb{E} M_0^2 = \mathbb{E} \langle M \rangle_{t \wedge S_n}$$

Take $t \to \infty$: $\mathbb{E} M_{t \wedge S_n}^2 \to \mathbb{E} M_{S_n}^2$ by dominated convergence

$\mathbb{E} \langle M \rangle_{t \wedge S_n} \to \mathbb{E} \langle M \rangle_{S_n}$ by monotone convergence

Take $n \to \infty$: $\mathbb{E} M_{S_n}^2 \to \mathbb{E} M_\infty^2$

$\mathbb{E} \langle M \rangle_{S_n} \to \mathbb{E} \langle M \rangle_\infty$

$$\Rightarrow \mathbb{E} \langle M \rangle_\infty = \mathbb{E} M_\infty^2 - \mathbb{E} M_0^2 = \mathbb{E}(M_\infty - M_0)^2 < \infty$$

2.5. Covariation

Defn. Let M, N be two continuous local martingales. Define

$$\langle M,N\rangle_t = \frac{1}{4}\left(\langle M+N\rangle_t - \langle M-N\rangle_t\right)$$

The process $\langle M,N\rangle$ is called the covariation or bracket of M and N.

Prop.

(i) $\langle M,N\rangle$ is the unique (up to distinguishability) finite variation process s.t. $M_tN_t - \langle M,N\rangle_t$ is a continuous local martingale.

(ii) The mapping $(M,N) \mapsto \langle M,N\rangle$ is bilinear and symmetric.

(iii)
$$\langle M,N\rangle_t = \lim_{n\to\infty} \langle M,N\rangle_t^{(n)}, \quad \text{where } \langle M,N\rangle_t^{(n)} = \sum_{i=1}^{[2^n t]} (M_{i2^{-n}} - M_{(i-1)2^{-n}})(N_{i2^{-n}} - N_{(i-1)2^{-n}})$$
with convergence ucp

(iv) For every stopping time T, $\langle M^T, N^T\rangle_t = \langle M^T, N\rangle = \langle M,N\rangle_{T\wedge t}$.

(v) If $M,N \in \mathcal{M}_c^2$ then $M_tN_t - \langle M,N\rangle_t$ is a UI martingale, and $(M-M_0, N-N_0)_{\mathcal{M}^2} = \mathbb{E}\langle M,N\rangle_\infty$

Proof. (i)

$$MN - \langle M,N\rangle = \frac{1}{4}\Big(\underbrace{(M+N)^2 - \langle M+N\rangle - (M-N)^2 - \langle M-N\rangle}_{\text{continuous local martingales}}\Big)$$

Uniqueness follows exactly as for quadratic variation.

(iii), (v) Similarly,

$$\langle M,N\rangle_t^{(n)} = \frac{1}{4}\left(\langle M+N\rangle_t^{(n)} - \langle M-N\rangle_t^{(n)}\right)$$

and the statements follow from the analogous statements for the quadratic variation.

(ii) Follows from (iii) or a uniqueness argument

(iv) By (iii),

$$\langle M^T, N^T\rangle_t = \langle M^T, N\rangle_t = \langle M,N\rangle_t \quad \text{on } \{T \geq t\}$$

$$\langle M^T, N^T\rangle_t - \langle M^T, N^T\rangle_s = \langle M^T, N\rangle_t - \langle M^T, N\rangle_s = 0 \quad \text{on } \{T \leq s < t\}$$

(v) Follows from case $M=N$.

<u>Example</u>. Let B and B' be two independent Brownian motions (adapted w.r.t. the same filtration). Then $\langle B, B'\rangle = 0$.

<u>Proof</u>. Assume $B_0 = B_0' = 0$. Then $X = \frac{1}{\sqrt{2}}(B+B')$ is a martingale (as a sum of two martingales) and X is a Brownian motion (check f.d. distribution). Thus $\langle X\rangle_t = \langle X, X\rangle_t = t$.

$$\Rightarrow \langle B, B'\rangle = \frac{1}{2}\left(\langle X, X\rangle - \frac{1}{2}\langle B, B\rangle - \frac{1}{2}\langle B', B'\rangle\right) = 0.$$

Prop. (Kunita – Watanabe). Let M, N be continuous local martingales and let H, K be two measurable processes. Then, almost surely,

$$\int_0^\infty |H_s|\,|K_s|\,|d\langle M,N\rangle_s| \leq \left(\int_0^\infty H_s^2\, d\langle M\rangle_s\right)^{\frac{1}{2}} \left(\int_0^\infty K_s^2\, d\langle N\rangle_s\right)^{\frac{1}{2}}. \quad \text{(KW)}$$

Proof. Write $\langle M,N\rangle_s^t = \langle M,N\rangle_t - \langle M,N\rangle_s$.

Claim: For all $0 \leq s < t$,

$$|\langle M,N\rangle_s^t| \leq \sqrt{\langle M,M\rangle_s^t}\,\sqrt{\langle N,N\rangle_s^t}. \qquad (*)$$

By continuity, we can assume that s,t are dyadic rationals. Then the claim follows from the approximation

$$\begin{aligned}
|\langle M,N\rangle_t^s| &\overset{\text{u.c.p}}{=} \lim_{n\to\infty} \left| \sum_{i=2^n s+1}^{2^n t} (M_{2^{-n}i} - M_{2^{-n}(i-1)})(N_{2^{-n}i} - N_{2^{-n}(i-1)}) \right| \\
&\overset{\text{CS}}{\leq} \lim_{n\to\infty} \left(\sum_{i=2^n s+1}^{2^n t} (M_{2^{-n}i} - M_{2^{-n}(i-1)})^2 \right)^{1/2} \left(\sum_{i=2^n s+1}^{2^n t} (N_{2^{-n}i} - N_{2^{-n}(i-1)})^2 \right)^{1/2} \\
&\overset{\text{u.c.p.}}{=} \left(\langle M,M\rangle_s^t\right)^{1/2} \left(\langle N,N\rangle_s^t\right)^{1/2}.
\end{aligned}$$

Now fix an event of probability 1 s.t. (*) holds for all $s<t$ (rational and by continuity then also for all $s<t$ real).

Claim: For all $0 \leq s < t$,

$$\int_s^t |d\langle M,N\rangle_u| \leq \sqrt{\langle M,M\rangle_s^t}\,\sqrt{\langle N,N\rangle_s^t}.$$

Indeed, for any subdivision $s=t_0<t_1<\dots<t_n=t$,

$$\sum_{i=1}^{n} \left|\langle M,N\rangle_{t_{i-1}}^{t_i}\right| \le \sum_{i=1}^{n} \sqrt{\langle M,M\rangle_{t_{i-1}}^{t_i}}\,\sqrt{\langle N,N\rangle_{t_{i-1}}^{t_i}}$$

$$\overset{CS}{\le} \left(\sum_{i=1}^{n} \langle M,M\rangle_{t_{i-1}}^{t_i}\right)^{1/2}\left(\sum_{i=1}^{n} \langle N,N\rangle_{t_{i-1}}^{t_i}\right)^{1/2} = \sqrt{\langle M,M\rangle_s^t}\,\sqrt{\langle N,N\rangle_s^t}.$$

and taking the sup over all subdivisions the claim follows.

Claim: For all bounded Borel sets $B\subset[0,\infty)$,

$$\int_B |d\langle M,N\rangle_u| = \sqrt{\int_B |d\langle M,M\rangle_u|}\,\sqrt{\int_B |d\langle N,N\rangle_u|}$$

For A a finite union of intervals, this follows from CS as above.

Exercise: Extend to all bounded Borel sets by a monotone class argument.

Claim: (KW) holds for $H=\sum h_\ell 1_{B_\ell}$, $K=\sum k_\ell 1_{B_\ell}$, (B_i) bounded Borel sets.

$$\int |H_sK_s|\,|d\langle M,N\rangle_s| = \sum_\ell |h_\ell k_\ell| \int_{B_\ell} |d\langle M,N\rangle_s|$$

$$\le \sum_\ell |h_\ell k_\ell| \left(\int_{B_\ell} d\langle M\rangle_s\right)^{1/2}\left(\int_{B_\ell} d\langle N\rangle_s\right)^{1/2}$$

$$\overset{CS}{\le} \left(\sum_\ell h_\ell^2 \int_{B_\ell} d\langle M\rangle_s\right)^{1/2}\left(\sum_\ell k_\ell^2 \int_{B_\ell} d\langle N\rangle_s\right)^{1/2} = \left(\int H_s^2\, d\langle M\rangle_s\right)^{1/2}\left(\int K_s^2\, d\langle N\rangle_s\right)^{1/2}$$

Finally, approximate general H,K by step functions as above.

2.6. Semimartingales

Defn. A (continuous) adapted process X is a (continuous) semimartingale if

$$X = X_0 + M + A$$

with $X_0 \in \mathcal{F}_0$, M a (continuous) local martingale with $M_0 = 0$, and A a (continuous) finite variation process with $A_0 = 0$.

The decomposition is unique up to indistinguishability.

Defn. Let $X = X_0 + M + A$ and $X' = X_0' + M' + A'$ be continuous semimartingales. Set

$$\langle X \rangle = \langle M \rangle, \quad \langle X, X' \rangle = \langle M, M' \rangle.$$

Prop.

$$\langle X, Y \rangle_t^{(n)} = \sum_{i=1}^{[2^n t]} \left(X_{i2^{-n}} - X_{(i-1)2^{-n}} \right)\left(Y_{i2^{-n}} - Y_{(i-1)2^{-n}} \right) \longrightarrow \langle X, Y \rangle_t \text{ u.c.p.}$$

Proof. Exercise.

3. The stochastic integral

3.1. Simple processes

Defn. The space of simple processes $\mathcal{E}$ consists of processes $H : \Omega \times [0,\infty)$ that can be written as

$$H_t(\omega) = \sum_{i=1}^{n} H_{i-1} \mathbb{1}_{(t_{i-1},t_i]}(t)$$

for $0 \leq t_0 < t_1 < \cdots < t_n$ and random variables $H_k \in \mathcal{F}_{t_k}$.

Defn. For $M \in \mathcal{M}^2$ and $H \in \mathcal{E}$, set

$$(H \cdot M)_t = \sum_{i=1}^{n} H_{i-1}(M_{t_i \wedge t} - M_{t_{i-1} \wedge t}).$$

Prop. If $M \in \mathcal{M}^2_c$ and $H \in \mathcal{E}$ then $H \cdot M \in \mathcal{M}^2_c$ and

$$\| H \cdot M \|_{\mathcal{M}^2} = \mathbb{E}\left(\int_0^\infty H_s^2 \, d\langle M \rangle_s \right). \qquad (*)$$

Proof. Claim: $H \cdot M$ is a martingale in $\mathcal{M}^2_c$

Let $X^i_t = H_{i-1}(M_{t_i \wedge t} - M_{t_{i-1} \wedge t})$. Then X^i is a martingale: for $s<t$,

$$\mathbb{E}(X^i_t \mid \mathcal{F}_s) = H_{i-1}\big(\underbrace{\mathbb{E}(M_{t \wedge t_i} \mid \mathcal{F}_s)}_{M_s} - M_{t_{i-1}}\big) = X^i_s \quad \text{for } t_{i-1} \leq s \leq t_i$$

$$\mathbb{E}(X^i_t \mid \mathcal{F}_s) = \mathbb{E}(H_{i-1}(0-0) \mid \mathcal{F}_s) = 0 = X^i_s \quad \text{for } s < t_{i-1}$$

$$\mathbb{E}(X^i_t \mid \mathcal{F}_s) = H_{i-1}(M_{t_i} - M_{t_{i-1}}) = X^i_s \quad \text{for } s > t_i$$

Moreover, $\|X^i\|_{\mathcal{M}^2} \leq 2 \|H\|_\infty \|M\|_{\mathcal{M}^2}$ so $X^i \in \mathcal{M}^2_c$.

Thus clearly also $H \cdot M = \sum_{i=1}^{n} X^i \in \mathcal{M}^2_c$.

Claim: (*) holds

The X^i are orthogonal: $E(X^i_\infty X^j_\infty) = E\Big(H_{i-1}(M_{t_i} - M_{t_{i-1}}) H_{j-1} E\big(\underbrace{M_{t_j} - M_{t_{j-1}} \mid \mathcal{F}_{t_{j-1}}}_{0}\big)\Big)$ $(j>i)$ and

$$\langle X^i\rangle_t = H^2_{i-1}\big(\langle M\rangle_{t_i\wedge t} - \langle M\rangle_{t_{i-1}\wedge t}\big).$$

$$\Rightarrow \mathbb{E}\langle H\cdot M, H\cdot M\rangle_\infty \overset{\uparrow}{=} \sum_{i=1}^n \mathbb{E}\langle X^i, X^i\rangle_\infty = \sum_{i=1}^n \mathbb{E}\Big(H^2_{i-1}\big(\langle M\rangle_{t_i} - \langle M\rangle_{t_{i-1}}\big)\Big)$$

Orthogonality and p. 31, (v)

$$= \mathbb{E}\Big(\int_0^\infty H_s^2 \, d\langle M\rangle_s\Big)$$

$$\Rightarrow \|H\cdot M\|^2_{M^2} = \mathbb{E}\langle H\cdot M\rangle_\infty = \mathbb{E}\Big(\int_0^\infty H_s^2 \, d\langle M\rangle_s\Big).$$

<u>Prop.</u> Let $M \in M^2_c$ and $H \in \mathcal{E}$. Then

$$\langle H\cdot M, N\rangle = H\cdot\langle M, N\rangle \quad \forall N \in M^2$$

<u>Proof.</u> Let $H\cdot M = \sum_{i=1}^n X^i$ as in the previous proof.

$$\langle X^i, N\rangle_t = H_{i-1}\langle M_{t_i\wedge\cdot} - M_{t_{i-1}\wedge\cdot}, N\rangle_t$$
$$= H_{i-1}\big(\langle M,N\rangle_{t_i\wedge t} - \langle M,N\rangle_{t_{i-1}\wedge t}\big)$$

$$\Rightarrow \langle H\cdot M, N\rangle_t = \int_0^t H_s \, d\langle M,N\rangle = (H\cdot\langle M,N\rangle)_t.$$

3.2. Itô isometry

Defn. Let $M \in \mathcal{M}_c^2$. Define $L^2(M)$ to be the space of (equivalence classes) of previsible $H : \Omega \times [0,\infty) \to \mathbb{R}$ s.t.

$$\|H\|_{L^2(M)} = \|H\|_M = \left(\mathbb{E}\left(\int_0^\infty H_s^2 \, d\langle M \rangle_s \right)^{1/2} \right) < \infty.$$

For $H, K \in L^2(M)$, set

$$(H,K)_{L^2(M)} = \mathbb{E}\left(\int_0^\infty H_s K_s \, d\langle M \rangle_s \right).$$

Fact. $L^2(M) = L^2(\Omega \times [0,\infty), \mathcal{P}, dP\, d\langle M \rangle)$ is a Hilbert space.

Prop. Let $M \in \mathcal{M}_c^2$. Then $\mathcal{E}$ is dense in $L^2(M)$.

Proof. Since $L^2(M)$ is a Hilbert space (complete!), it suffices to show that if $(K,H) = 0 \ \forall H \in \mathcal{E}$ then $K = 0$.

Assume that $(K,H) = 0 \ \forall H \in \mathcal{E}$, and set

$$X_t = \int_0^t K_s \, d\langle M \rangle_s .$$

This makes sense since (use $K \in L^2(M)$, $M \in \mathcal{M}^2$)

$$\mathbb{E}\left(\int_0^t |K_s| \, d\langle M \rangle_s \right) \overset{CS}{\leq} \left(\mathbb{E}\left(\int_0^t |K_s|^2 \, d\langle M \rangle_s \right) \right)^{1/2} \left(\mathbb{E} \langle M \rangle_\infty \right)^{1/2} < \infty$$

$\Rightarrow$ X is well-defined finite variation process, $X_t \in L^1 \ \forall t$

Claim: X is a continuous martingale.

Let $0 \le s < t$, $F \in \mathcal{F}_s$, $H = F \mathbb{1}_{(s,t]} \in \mathcal{E}$, F bounded. By assumption, then

$$0 = (K, H) = \mathbb{E}\left(F \int_s^t K_u \, d\langle M \rangle_u\right)$$

$\Rightarrow$ $\mathbb{E}\left(F(X_t - X_s)\right) = 0 \quad \forall s < t,\ F \in \mathcal{F}_s$ bounded

$\Rightarrow$ $\mathbb{E}(X_t \mid \mathcal{F}_s) = X_s$, i.e., X is a continuous martingale.

Thus, X is a finite variation continuous martingale, so $X = 0$.

$\Rightarrow$ $K = 0 \quad d\langle M \rangle$ a.e.

$\Rightarrow$ $K = 0$ in $L^2(M)$.

<u>Thm</u>. Let $M \in \mathcal{M}^2_c$. Then:

(i) The map $H \in \mathcal{E} \mapsto H \cdot M \in \mathcal{M}^2_c$ extends uniquely to an isometry $L^2(M) \to \mathcal{M}^2_c$ (the <u>Itô isometry</u>).

(ii) $H \cdot M$ is the unique martingale in $\mathcal{M}^2_c$ s.t.

$$\underset{\uparrow \text{stoch. integral}}{\langle H \cdot M, N \rangle} = \underset{\uparrow \text{fin. var. integral}}{H \cdot \langle M, N \rangle} \qquad \forall N \in \mathcal{M}^2_c.$$

(iii) If T is a stopping time then

$$(\mathbb{1}_{[0,T]} H) \cdot M = (H \cdot M)^T = H \cdot M^T.$$

Defn. $H \bullet M$ is the stochastic integral of H w.r.t. M and we write

$$(H \bullet M)_t = \int_0^t H_s \, dM_s .$$

Proof. (i) For $H \in \mathcal{E}$, we have already seen that

$$\| H \cdot M \|_{\mathcal{M}^2}^2 = \mathbb{E}\left(\int_0^\infty H_s^2 \, d\langle M \rangle_s \right) = \| H \|_{L^2(M)}^2 .$$

Since $\mathcal{E} \subset L^2(M)$ is dense and $\mathcal{M}_c^2$ is a Hilbert space, it follows that the map $H \mapsto H \bullet M$ extends uniquely to all of $L^2(M)$ and the extension is also an isometry.

(ii) We have already seen that $\langle H \bullet M, N \rangle = H \bullet \langle M, N \rangle$ holds for $H \in \mathcal{E}$. Given $H \in L^2(M)$, choose $(H^n) \subset \mathcal{E}$ s.t. $H^n \to H$. Then $H^n \cdot M \to H \bullet M$ in $\mathcal{M}_c^2$. We will justify

$$\langle H \cdot M, N \rangle_\infty \overset{(\dagger)}{=} \lim_{n \to \infty} \langle H^n \cdot M, N \rangle_\infty \quad \text{in } L^1$$

$$= \lim_{n \to \infty} \left(H^n \cdot \langle M, N \rangle \right)_\infty$$

$$\overset{(\dagger)}{=} \left(H \cdot \langle M, N \rangle \right)_\infty . \quad \text{in } L^1$$

Here (†) hold by the Kunita-Watanabe inequality:

$$\mathbb{E}|\langle H\cdot M - H^n\cdot M, N\rangle_\infty| \le \left(\mathbb{E}\,\langle H\cdot M - H^n\cdot M\rangle_\infty\right)^{1/2} \mathbb{E}\left(\langle N\rangle_\infty\right)^{1/2}$$

$$= \underbrace{\|H\cdot M - H^n\cdot M\|_{M^2}}_{\to 0} \|N\|_{M^2}$$

$$\mathbb{E}\left((H-H^n)\cdot\langle M,N\rangle\right)_\infty \le \underbrace{\|H-H^n\|_{L^2(M)}}_{\to 0} \|N\|_{M^2}$$

Thus $\langle H\cdot M, N\rangle_\infty = (H\cdot\langle M,N\rangle)_\infty$.

Replacing N by N^t gives $\langle H\cdot M, N\rangle_t = (H\cdot\langle M,N\rangle)_t$.

stopped martingale

Uniqueness: if $X\in M_c^2$ has the same property as $H\cdot M$ then

$$\langle H\cdot M - X, N\rangle = 0 \quad \forall N\in M_c^2$$

$\Rightarrow$ $H\cdot M = X$ in M_c^2. (by taking $N = H\cdot M - X$ and using that a mart $X\in M_c^2$ in M_c^2 vanishes iff $\langle X\rangle = 0$).

(iii) For $N\in M^2$,

$$\langle (H\cdot M)^T, N\rangle_t = \langle H\cdot M, N\rangle_{t\wedge T} \overset{(ii)}{=} (H\cdot\langle M,N\rangle)_{t\wedge T} = (H\mathbb{1}_{[0,T]}\cdot\langle M,N\rangle)_t$$

$\Rightarrow$ $(H\cdot M)^T = (\mathbb{1}_{[0,T]}H)\cdot M$ by (ii)

$$\langle H\cdot M^T, N\rangle_t \overset{(ii)}{=} H\cdot\langle M^T,N\rangle_t = (H\cdot\langle M,N\rangle)_{t\wedge T} = (\mathbb{1}_{[0,T]}H\cdot\langle M,N\rangle)$$

$\Rightarrow$ $H\cdot M^T = (\mathbb{1}_{[0,T]}H)\cdot M$ by (ii)

Rk. (ii) can be written as

$$\left\langle \int_0^{\cdot} H_s\, dM_s, N \right\rangle_t = \langle H\bullet M, N\rangle_t = \left(H\bullet\langle M,N\rangle\right)_t = \int_0^t H_s\, d\langle M,N\rangle_s$$

i.e. "the integral commutes with the bracket".

Cor.

$$\langle H\bullet M, K\bullet N\rangle = H\bullet\langle M, K\bullet N\rangle = H\bullet\left(K\bullet\langle M,N\rangle\right) \underset{\uparrow}{=} (HK)\bullet\langle M,N\rangle$$

associativity of fin. var. int.
$d\left(K\bullet\langle M,N\rangle\right)_s = K_s\, d\langle M,N\rangle_s$

i.e. $\left\langle \int_0^{\cdot} H_s\, dM_s , \int_0^{\cdot} K_s\, dN_s \right\rangle_t = \int_0^t H_s K_s\, d\langle M,N\rangle_s$

Cor.

$$\mathbb{E}\left(\int_0^t H_s\, dM_s\right) = 0$$

$$\mathbb{E}\left(\left(\int_0^t H_s\, dM_s\right)\left(\int_0^t K_s\, dN_s\right)\right) = \int_0^t H_s K_s\, d\langle M,N\rangle_s$$

Proof. $H\bullet M$ and $(H\bullet M)(K\bullet N) - \langle H\bullet M, K\bullet N\rangle$ are martingales starting at 0.

Cor.

$$\mathbb{E}\left(\int_0^t H_s\, dM_s \,\middle|\, \mathcal{F}_s\right) = \int_0^s H_s\, dM_s$$

Proof. $H\bullet M$ is a martingale.

Cor. Let $H \in L^2(M)$. Then $KH \in L^2(M)$ iff $K \in L^2(H \cdot M)$ and then

$$(KH) \cdot M = K \cdot (H \cdot M).$$

Proof.

$$\mathbb{E}\left(\int_0^\infty K_s^2 H_s^2 \, d\langle M_s \rangle\right) \overset{\text{above}}{=} \mathbb{E}\left(\int_0^\infty K_s^2 \, d\langle H \cdot M \rangle_s\right) \text{ so } KH \in L^2(M) \Leftrightarrow K \in L^2(H \cdot M).$$

For $N \in \mathcal{M}_c^2$,

$$\langle (KH) \cdot M, N \rangle_t = (KH \cdot \langle M, N \rangle)_t$$

$$= \int_0^t \underbrace{K_s H_s \, d\langle M, N \rangle_s}_{d\int_0^\cdot H_u \, d\langle M,N\rangle_u = d(H \cdot \langle M, N \rangle)} = (K \cdot (H \cdot \langle M, N \rangle))_t =$$

$$\langle K \cdot (H \cdot M), N \rangle_t = K \cdot \langle H \cdot M, N \rangle = K \cdot (H \cdot \langle M, N \rangle)$$

$\Rightarrow (KH) \cdot M = K \cdot (H \cdot M)$ by uniqueness.

3.3. Extension to local martingales

Defn. Let $L^2_{loc}(M)$ be the space of previsible H s.t. a.s.

$$\int_0^t H_s^2 \, d\langle M\rangle_s < \infty \quad \forall t > 0.$$

Thm. Let M be a continuous local martingale.

(i) For every $H \in L^2_{loc}(M)$, there is a unique continuous local martingale $H \cdot M$ with $(H \cdot M)_0 = 0$ s.t.

$$\langle H \cdot M, N\rangle = H \cdot \langle M, N\rangle \quad \forall N \text{ continuous local martingale.}$$

(ii) If T is a stopping time,

$$(1_{[0,T]} H) \cdot M = (H \cdot M)^T = H \cdot M^T$$

(iii) If $H \in L^2_{loc}(M)$ and K is previsible then $K \in L^2_{loc}(H \cdot M)$ iff $HK \in L^2_{loc}(M)$ and then

$$H \cdot (K \cdot M) = HK \cdot M.$$

Finally, if $M \in \mathcal{M}_c^2$ and $H \in L^2(M)$ then the defn. is the same as before.

Proof. (i) Assume $M_0 = 0$ and (setting $H = 0$ where this fails)

$$\int_0^t H_s^2 \, d\langle M\rangle_s < \infty \quad \forall t \geq 0 \; \forall \omega \in \Omega.$$

Set

$$S_n = \inf\{t \geq 0 : \int_0^t (1 + H_s^2) \, d\langle M\rangle_s \geq n\}.$$

Note that S_n are stopping times with $S_n \uparrow \infty$.

$$\langle M^{S_n}, M^{S_n}\rangle_t = \langle M, M\rangle_{t \wedge S_n} \leq n$$

$\Rightarrow M^{S_n} \in \mathcal{M}_c^2$ and

$$\int_0^\infty H_s^2 \, d\langle M^{S_n}\rangle_s = \int_0^{S_n} H_s^2 \, d\langle M\rangle_s \leq n.$$

$\Rightarrow H \in L^2(M^{S_n})$ and $H \cdot M^{S_n}$ is defined.

$(H \cdot M^{S_n}) = (H \cdot M^{S_m})^{S_n}$ for $m > n$ (stopping times commute with $\cdot$)

$\Rightarrow$ there is a unique process denoted $H \cdot M$ s.t.
$(H \cdot M)^{S_n} = H \cdot M^{S_n} \quad \forall n$

$H \cdot M$ is adapted, has continuous sample paths and is a local martingale since $(H \cdot M)^{S_n}$ are martingales.

Claim: $\langle H\cdot M, N\rangle = H\cdot\langle M,N\rangle$

Assume that $N_0=0$. Set

$$S_n' = \inf\{t\geq 0 : |N_t|\geq n\},\quad T_n = S_n\wedge S_n'.$$

$\Rightarrow$ $N^{S_n'}\in \mathcal{M}_c^2$ and

$$\begin{aligned}\langle H\cdot M, N\rangle^{T_n} &= \langle (H\cdot M)^{S_n}, N^{S_n'}\rangle \\ &= \langle H\cdot M^{S_n}, N^{S_n'}\rangle \\ &= H\cdot\langle M^{S_n}, N^{S_n'}\rangle \\ &= H\cdot\langle M,N\rangle^{T_n} \\ &= (H\cdot\langle M,N\rangle)^{T_n}\end{aligned}$$

Since $T_n \uparrow \infty$ thus $\langle H\cdot M, N\rangle = H\cdot\langle M,N\rangle$. Uniqueness follows as before.

(ii),(iii) follow as in proof for $M\in\mathcal{M}_c^2$, $H\in L^2(M)$ since it only uses property (i)

If $M\in\mathcal{M}_c^2$, $H\in L^2(M)$ then $\langle H\cdot M, H\cdot M\rangle = H^2\cdot\langle M,M\rangle$ by (i)

$\Rightarrow$ $H\cdot M\in\mathcal{M}_c^2$

Equivalent of (i) from previous theorem shows that $H\cdot M$ is consistent with previous defn.

3.4. Extension to semimartingales

<u>Defn.</u> A previsible process H is <u>locally bounded</u> if

$$\forall t \geq 0: \sup_{s \leq t} |H_s| < \infty \quad \text{a.s.}$$

<u>Fact.</u> • Any adapted continuous process is locally bounded.

• If H is locally bounded and V a finite variation process then

$$\forall t \geq 0: \int_0^t |H_s| |dV_s| < \infty \quad \text{a.s.}$$

• If H is locally bounded and M a continuous local martingale then $H \in L^2_{loc}(M)$.

<u>Defn.</u> Let $X = X_0 + M + A$ be a continuous semimartingale, H a locally bounded process. Then the <u>stochastic integral</u> $H \cdot X$ is the continuous semimartingale

$$H \cdot X = H \cdot M + H \cdot A$$

Lebesgue-Stieltjes integral

Itô integral

and we write

$$(H \cdot X)_t = \int_0^t H_s \, dX_s .$$

Prop.

(i) $(H,X) \mapsto H\cdot X$ is bilinear

(ii) $H\cdot(K\cdot X) = (HK)\cdot X$ if H,K are locally bounded

(iii) $(H\cdot X)^T = H 1_{[0,T]} \cdot X = H\cdot X^T$ for every stopping time T

(v) If X is a continuous local martingale (resp. a finite variation process), then so is $H\cdot X$

(iv) If $H = \sum_{i=1}^n H_{i-1} 1_{(t_{i-1},t_i]}$ and $H_{i-1} \in \mathcal{F}_{t_{i-1}}$ (not nec. bounded),

$$(H\cdot X)_t = \sum_{i=1}^n H_{i-1}(X_{t_i\wedge t} - X_{t_{i-1}\wedge t}).$$

Proof. (i)-(v) follow from analogous properties for continuous local martingales and finite variation processes.

(iv) is also clear for the finite variation part of X. Assume $X \in \mathcal{M}^2_c$. Then (iv) is true by defn if the H_{i-1} are bounded. Set

$$T_n = \inf\{t \geq 0 : |H_t| \geq n\}$$

Then T_n is a stopping time, $T_n \uparrow \infty$, and $H 1_{[0,T_n]} \in \mathcal{E}$. Thus

$$(H\cdot X)_{t\wedge T_n} = \sum_{i=1}^n H_{i-1} 1_{[0,T_n]} (X_{t_i\wedge t} - X_{t_{i-1}\wedge t})$$

Since $T_n \uparrow \infty$ the claim follows.

Prop. (Stochastic DCT). Let X be a continuous semimartingale. Let H be previsible and locally bounded, and let K be previsible and nonnegative. Let $t>0$. Assume that a.s.

(i) $H_s^n \xrightarrow{n\to\infty} H_s$ for all $s\in[0,t]$;

(ii) $|H_s^n| \le K_s$ for all $s\in[0,t]$ and $n\in\mathbb{N}$;

(iii) $\int_0^t (K_s)^2\, d\langle M\rangle_s < \infty$ and $\int_0^t K_s\, |dA_s| < \infty$ where A is the finite variation part of X (both cond. are ok if K is locally bounded)

Then $\int_0^t H_s^n\, dX_s \to \int_0^t H_s\, dX_s$ u.c.p.

Proof. Let $X = X_0 + M + A$. For the finite variation part A, the claim follows from the usual DCT. Set

$$T_m = \inf\left\{t\ge 0 : \int_0^t (K_s)^2\, d\langle M\rangle_s \ge m\right\}.$$

Then

$$\mathbb{E}\left(\left(\int_0^{T_m\wedge t} H_s^n\, dM_s - \int_0^{T_m\wedge t} H_s\, dM_s\right)^2\right) \le \mathbb{E}\left(\int_0^{T_m\wedge t} (H_s^n - H_s)^2\, d\langle M\rangle_s\right) \to 0$$

DCT, (i), (ii), $\int_0^{T_m\wedge t} K_s\, d\langle M\rangle_s \le m$

Since $T_m\wedge t = t$ eventually, $\mathbb{P}(T_m\wedge t = t)\to 1$ the result follows in the sense of convergence in prob.

To get u.c.p. convergence, use Doob's inequality on the LHS.

Prop. Let X be a continuous semimartingale and let H be an adapted bounded left-continuous process. Then, for every subdivision $0 = t_0^{(m)} < t_1^{(m)} < \cdots < t_{n_m}^{(m)} = t^{(m)}$ of $[0,t]$ with $\max_i |t_i - t_{i-1}| \xrightarrow{m\to\infty} 0$,

$$\lim_{m\to\infty} \sum_{i=1}^{n_m} H_{t_{i-1}^n} \left(X_{t_i^n} - X_{t_{i-1}^n} \right) = \int_0^t H_s \, dX_s. \quad \text{u.c.p.}$$

Proof. This follows by dominated convergence as it did for finite variation processes (p. 10).

3.5. Itô formula

Thm (Integration by parts). Let X, Y be continuous semimartingales. Then a.s.

$$X_t Y_t - X_0 Y_0 = \int_0^t X_s \, dY_s + \int_0^t Y_s \, dX_s + \underbrace{\langle X, Y \rangle_t}_{\text{Itô correction (absent if } X,Y \text{ are f.v.)}}$$

Proof. Clearly,

$$X_t Y_t - X_s Y_s = X_s (Y_t - Y_s) + (X_t - X_s) Y_s + (X_t - X_s)(Y_t - Y_s)$$

Thus

$$X_{k2^{-n}} Y_{k2^{-n}} - X_0 Y_0 = \sum_{i=1}^{k} \left(X_{i2^{-n}} Y_{i2^{-n}} - X_{(i-1)2^{-n}} Y_{(i-1)2^{-n}} \right)$$

$$= \sum_{i=1}^{k} \Big(X_{(i-1)2^{-n}} \left(Y_{i2^{-n}} - Y_{(i-1)2^{-n}} \right) + \left(X_{i2^{-n}} - X_{(i-1)2^{-n}} \right) Y_{(i-1)2^{-n}} + \left(X_{i2^{-n}} - X_{(i-1)2^{-n}} \right)\left(Y_{i2^{-n}} - Y_{(i-1)2^{-n}} \right) \Big)$$

For $t \in 2^{-m}\mathbb{N}$, $m < n$, (with u.c.p. convergence),

$$\left.\begin{aligned} \sum_{i=1}^{2^n t} X_{(i-1)2^{-n}} \left(Y_{i2^{-n}} - Y_{(i-1)2^{-n}} \right) &\longrightarrow (X \cdot Y)_t \\ \sum_{i=1}^{2^n t} \left(X_{i2^{-n}} - X_{(i-1)2^{-n}} \right) Y_{(i-1)2^{-n}} &\longrightarrow (Y \cdot X)_t \end{aligned}\right\} \text{ last section}$$

$$\sum_{i=1}^{2^n t} \left(X_{i2^{-n}} - X_{(i-1)2^{-n}} \right)\left(Y_{i2^{-n}} - Y_{(i-1)2^{-n}} \right) \longrightarrow \langle X, Y \rangle_t \quad \Big\} \text{ definition of quadratic var.}$$

This implies the claim for dyadic rationals. For $t \in \mathbb{R}$ use continuity.

Thm (Itô's formula). Let $X^1, \dots, X^p$ be continuous semimartingales, and let $f : \mathbb{R}^p \to \mathbb{R}$ be in C^2. Then, with $X = (X^1, \dots, X^p)$, a.s.

$$f(X_t) = f(X_0) + \underbrace{\sum_{i=1}^{p} \int_0^t \frac{\partial f}{\partial x^i}(X_s)\, dX^i_s}_{\text{finito variation}} + \underbrace{\frac{1}{2} \sum_{i,j=1}^{p} \int_0^t \frac{\partial^2 f}{\partial x^i \partial x^j}(X_s)\, d\langle X^i, X^j\rangle_s}_{\text{cont. local martingale}} \qquad (*)$$

Proof. Claim: $(*)$ holds when f is a polynomial.

For f constant, $(*)$ is obvious.

Suppose that $(*)$ holds for some f. Apply IBP to $g(x) = x^k f(x)$: (x^k: k-th component)

$$g(X_t) = g(X_0) + \underbrace{\int_0^t X^k_s\, df(X_s)}_{\sum_{i=1}^{p} \int_0^t X^k_s \frac{\partial f}{\partial x^i}(X_s)\, dX^i_s + \frac{1}{2}\sum_{i,j=1}^{p} \int_0^t X^k_s \frac{\partial^2 f}{\partial x^i \partial x^j}(X_s)\, d\langle X^i, X^j\rangle_s} + \int_0^t f(X_s)\, dX^k_s + \underbrace{\langle X^k, f(X)\rangle_t}_{\sum_{i=1}^{p} \int_0^t \frac{\partial f}{\partial x^i}(X_s)\, d\langle X^k, X^i\rangle_s}$$

(using $H \cdot (K \cdot X) = (HK) \cdot X$
$\langle Y, H \cdot X\rangle = H \cdot \langle X, Y\rangle$)

$$\Rightarrow g(X_t) = g(X_0) + \sum_{i=1}^{p} \int_0^t \frac{\partial g}{\partial x^i}(X_s)\, dX^i_s + \frac{1}{2} \sum_{i,j=1}^{p} \int_0^t \frac{\partial^2 g}{\partial x^i \partial x^j}(X_s)\, d\langle X^i, X^j\rangle_s$$

Thus $(*)$ holds for all polynomials f.

Write $X = X_0 + M + A$.

Claim: $(*)$ holds for all $f \in C^2$ if $|X_t(\omega)| \le n$, $\int_0^t |dA_s| \le n$ $\forall (t, \omega)$.

By the Weierstraß approximation theorem, there are polynomials p_k s.t.

$$\sup_{|x| \le k} \left(|f(x) - p_k(x)| + \max_i \left| \frac{\partial f}{\partial x^i}(x) - \frac{\partial p_k}{\partial x^i}(x) \right| + \max_{ij} \left| \frac{\partial^2 f}{\partial x^i \partial x^j}(x) - \frac{\partial^2 p_k}{\partial x^i \partial x^j}(x) \right| \right) \le \frac{1}{k}.$$

Taking limits, in probability,

$$f(X_t) - f(X_0) = \lim_{k\to\infty} \left(p_k(X_t) - p_k(X_0)\right)$$

$$\int_0^t \frac{\partial f}{\partial x^i}(X_s)\, dX_s^i = \lim_{k\to\infty} \int_0^t \frac{\partial p_k}{\partial x^i}(X_s)\, dX^i \quad \text{by stochastic DCT}$$

$$\int_0^t \frac{\partial^2 f}{\partial x^i \partial x^j}(X_s)\, d\langle X^i, X^j\rangle_s = \lim_{k\to\infty} \int_0^t \frac{\partial^2 p_k}{\partial x^i \partial x^j}\, d\langle X^i, X^j\rangle \quad \text{by DCT}$$

Claim: $(*)$ holds without restriction.

Let $T_n = \inf\{t \geq 0 : |X_t| \geq n, \int_0^t |dA_s| \geq n\}$. Then, by the above,

$$f(X_t^{T_n}) = f(X_0) + \sum_{i=1}^p \int_0^t \frac{\partial f}{\partial x^i}(X_s^{T_n})\, d(X^i)_s^{T_n} + \frac{1}{2}\sum_{i,j=1}^p \int_0^t \frac{\partial^2 f}{\partial x^i \partial x^j}(X_s^{T_n})\, d\langle (X^i)^{T_n}, (X^j)^{T_n}\rangle_s$$

$$= f(X_0) + \sum_{i=1}^p \int_0^{t\wedge T_n} \frac{\partial f}{\partial x^i}(X_s)\, dX_s^i + \frac{1}{2}\sum_{i,j=1}^p \int_0^{t\wedge T_n} \frac{\partial^2 f}{\partial x^i \partial x^j}(X_s)\, d\langle X^i, X^j\rangle_s .$$

Take $T_n \to \infty$.

<u>Example</u>. Let B be a standard Brownian motion, $B_0 = 0$, $f(x) = x^2$.

$\Rightarrow B_t^2 = 2\int_0^t B_s\, dB_s + t \Rightarrow B_t^2 - t = 2\int_0^t B_s\, dB_s$ is a cont. loc. mart.

<u>Example</u>. Let $B = (B^1, \dots, B^d)$ be a standard d-dim. BM.

$$\Rightarrow f(t, B_t) - f(0, B_0) - \int_0^t \left(\frac{\partial}{\partial s} + \frac{1}{2}\Delta\right) f(s, B_s)\, ds$$

$$= \sum_{i=1}^d \int_0^t \frac{\partial}{\partial x^i} f(s, B_s)\, dB_s^i \text{ is a cont. loc. mart.}$$

(Apply Itô with $X = (t, B_t^1, \dots, B_t^d)$.)

Rk. Itô's formula (*) is often stated in differential form as

$$df(X_t) = \sum_i \frac{\partial f}{\partial x^i}\, dX^i + \frac{1}{2}\sum_{i,j} \frac{\partial^2 f}{\partial x^i \partial x^j}\, d\langle X^i, X^j\rangle .$$

It is the chain rule for stochastic calculus.

In the case of BM,

$$df(B_t) = f'(B_t)\, dB_t + \frac{1}{2} f''(B_t)\, dt$$

Formally, one expands f using that "$(dB)^2 = dt$" and "$(dt)^2 = 0$".

The following formal computational rules hold:

$$Z_t - Z_0 = \int_0^t H_s\, dX_s \iff dZ_t = H_t\, dX_t$$

$$Z_t - \langle X, Y\rangle_t = \int_0^t d\langle X,Y\rangle_s \iff dZ_t = dX_t\, dY_t$$

Then:

- $H_t(K_t\, dX_t) = (H_t K_t)\, dX_t$
- $H_t(dX_t\, dY_t) = (H_t\, dX_t)\, dY_t$
- $d(X_t Y_t) = X_t\, dY_t + Y_t\, dX_t + dX_t\, dY_t$
- $df(X_t) = \sum_i \frac{\partial f}{\partial x^i}(X_t)\, dX_t^i + \frac{1}{2}\sum_{i,j} \frac{\partial^2 f}{\partial x^i \partial x^j}(X_t)\, dX_t^i\, dX_t^j .$

4. Applications to Brownian motion and martingales

4.1. Brownian motion: Lévy's characterisation, Dubins-Schwarz Theorem

Thm. Let $X = (X^1, \dots, X^d)$ be continuous local martingales. Suppose that $X_0 = 0$ and that $\langle X^i, X^j \rangle_t = \delta_{ij} t$ for all i, j, $t \geq 0$. Then X is a standard d-dimensional Brownian motion.

Proof. Let $0 \leq s < t$. It suffices to show that $X_t - X_s$ is independent of $\mathcal{F}_s$ and that $X_t - X_s \sim \mathcal{N}(0, (t-s)\, id_{d \times d})$.

Claim: $\mathbb{E}\left(e^{i\theta \cdot (X_t - X_s)} \mid \mathcal{F}_s\right) = e^{-\frac{1}{2}|\theta|^2 (t-s)}$ for all $\theta \in \mathbb{R}^d$, $s < t$,

which implies both claims

Fix $\theta \in \mathbb{R}^d$ and set $Y_t = \theta \cdot X_t = \sum_{j=1}^d \theta^j X_t^j$.

$\Rightarrow \langle Y \rangle_t = \langle Y, Y \rangle_t = \sum_{j,k=1}^d \theta^j \theta^k \langle X^j, X^k \rangle_t = |\theta|^2 t$ by assumption.

Let $Z_t = e^{iY_t + \frac{1}{2}\langle Y \rangle_t} = e^{i\theta \cdot X_t + \frac{1}{2}|\theta|^2 t}$.

By Itô's formula with $X = iY + \frac{1}{2}\langle Y \rangle$, $f(x) = e^x$,

$$dZ_t = Z_t \left(i\, dY_t - \tfrac{1}{2} d\langle Y \rangle_t + \tfrac{1}{2} d\langle Y \rangle_t \right) = i Z_t\, dY_t$$

$\Rightarrow Z$ is a continuous local martingale

Z is bounded on every interval $[0,t]$ $\Rightarrow$ Z is a martingale, $Z_0 = 1$.

$\Rightarrow \mathbb{E}(Z_t \mid \mathcal{F}_s) = Z_s$

$\Rightarrow \mathbb{E}\left(e^{i\theta\cdot(X_t - X_s)} \mid \mathcal{F}_s\right) = e^{-\frac{1}{2}|\theta|^2(t-s)}$.

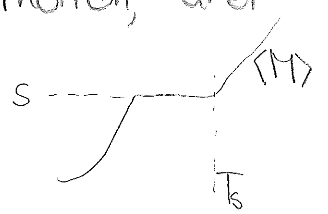
The theorem is called Levy's characterisation of Brownian motion.

right-continuous inverse of $\langle M \rangle$

Thm (Dubins–Schwarz). Let M be a continuous local martingale with $M_0 = 0$ and $\langle M \rangle_\infty = \infty$ a.s. Let $T_s = \inf\{t \geq 0 : \langle M \rangle_t > s\}$. Let $B_s = M_{T_s}$, $\mathcal{G}_s = \mathcal{F}_{T_s}$. Then T_s is an $(\mathcal{F}_t)$-stopping time, $\langle M \rangle_{T_s} = s$ for all $s \geq 0$, B is a $(\mathcal{G}_s)$ Brownian motion, and

$$M_t = B_{\langle M \rangle_t}.$$

M is a (random) time change of B



Proof $\langle M \rangle$ continuous, adapted, $\langle M \rangle_\infty = \infty$ a.s. $\Rightarrow T_s$ is a stopping time and $T_s < \infty$ for all $s \geq 0$, a.s.

Redefine $T_s = 0$ if $\langle M \rangle_\infty < \infty$. T_s is still a stopping time.

Claim: $(\mathcal{G}_s)$ is a filtration obeying the usual conditions, $\mathcal{G}_\infty = \mathcal{F}_\infty$.

Indeed, if $A \in \mathcal{G}_s$, then $A \cap \{T_t \leq u\} = A \cap \{T_s \leq u\} \cap \{T_t \leq u\} \in \mathcal{F}_u$

$\Rightarrow A \in \mathcal{F}_{T_t} = \mathcal{G}_t$, for any $t \geq s$.

$\Rightarrow (\mathcal{G}_s)$ is a filtration.

Right-continuity of $(\mathcal{G}_s)$ follows from that of $(\mathcal{F}_t)$ and right-continuity of T_s.

<u>Claim:</u> B is adapted to $(\mathcal{G}_s)$

If X is càdlàg and T a stopping time, then $X_T 1_{\{T<\infty\}} \in \mathcal{F}_T$.
($\to$ Advanced Probability).

Apply this with $X = M$, $T = T_s$, $\mathcal{F}_T = \mathcal{G}_s$: Thus $B_s \in \mathcal{G}_s$.

<u>Claim:</u> B is continuous

T_s is increasing, càdlàg in s $\Rightarrow$ $B_s = M_{T_s}$ is càdlàg and thus right-continuous

B is left-continuous $\Leftrightarrow$ $B_s = B_{s-}$ $\Leftrightarrow$ $M_{T_s} = M_{T_{s-}}$,

$$T_{s-} = \inf\{t \geq 0 : \langle M \rangle_t = s\}.$$

If $T_s = T_{s-}$ there is nothing to show. Assume $T_s > T_{s-}$.

$\Rightarrow$ $\langle M \rangle$ is constant on $[T_{s-}, T_s]$.

<u>Lemma</u> M is constant on $[a,b]$ $\Leftrightarrow$ $\langle M \rangle$ is constant on $[a,b]$ for all $a<b$, a.s.

Thus if $T_s > T_{s-}$ we also have $M_{T_s} = M_{T_{s-}}$, i.e. B is continuous.

<u>Claim:</u> B is a Brownian motion.

Let $0 \leq r < s$.

$\langle M^{T_s} \rangle_\infty = \langle M \rangle_{T_s} = s \Rightarrow M^{T_s} \in \mathcal{M}_c^2 \Rightarrow (M^2 - \langle M \rangle)^{T_s}$ is a UI martingale.

OST implies

$$\mathbb{E}(B_s \mid \mathcal{G}_r) = \mathbb{E}(M_\infty^{T_s} \mid \mathcal{F}_{T_r}) = M_{T_r} = B_r$$

$$\mathbb{E}(B_s^2 - s \mid \mathcal{G}_r) = \mathbb{E}\big((M^2 - \langle M\rangle)^{T_s} \mid \mathcal{F}_{T_r}\big) = M_{T_r}^2 - \langle M\rangle_{T_r} = B_r^2 - r$$

$\Rightarrow$ B is a continuous martingale with $\langle B\rangle_s = s$ for all $s \geq 0$.

$\Rightarrow$ B is a $(\mathcal{G}_s)$ - Brownian motion by Lévy's characterisation.

Proof of lemma By continuity, it suffices to prove for any fixed $a<b$ that

$$\{M_t = M_a \ \forall t \in [a,b]\} = \{\langle M\rangle_b = \langle M\rangle_a\} \quad \text{a.s.}$$

Claim: $\{M \text{ is const. on } [a,b]\} \subseteq \{\langle M\rangle_b = \langle M\rangle_a\}$ a.s.

$N_t := M_t - M_{t\wedge a}$ satisfies $\langle N\rangle_t = \langle M\rangle_t - \langle M\rangle_{t\wedge a}$.

Let $T_\varepsilon = \inf\{t \geq 0 : \langle N\rangle_t \geq \varepsilon\}$.

$\Rightarrow$ $N^{T_\varepsilon} \in \mathcal{M}_c^2$ since $\langle N^{T_\varepsilon}\rangle_t \leq \varepsilon$,

$$\mathbb{E} N_{t\wedge T_\varepsilon}^2 = \mathbb{E}\langle N\rangle_{t\wedge T_\varepsilon} \leq \varepsilon$$

$$\Rightarrow \mathbb{E}\left(1_{\{\langle M\rangle_b = \langle M\rangle_a\}} N_t^2\right) = \mathbb{E}\left(1_{\{\langle M\rangle_b = \langle M\rangle_a\}} N_{t\wedge T_\varepsilon}^2\right) \leq \varepsilon \quad \forall \varepsilon > 0$$

$\Rightarrow$ $N_t = 0$ a.s. on $\{\langle M\rangle_b = \langle M\rangle_a\}$.

Other direction: exercise (use approximation for example).

4.2. Girsanov's Theorem

Example. Let $X \sim \mathcal{N}(0,C)$ be an n-dimensional centred Gaussian vector with positive definite covariance matrix $C = (C_{ij})_{i,j=1}^{n}$:

$$\mathbb{E}(f(X)) = \det\left(\frac{M}{2\pi}\right)^{1/2} \int_{\mathbb{R}^n} f(x)\, e^{-\frac{1}{2}(x, Mx)}\, dx, \quad M = C^{-1}$$

Let $a \in \mathbb{R}^n$. Then

$$\mathbb{E}(f(X+a)) = \det\left(\frac{M}{2\pi}\right)^{1/2} \int_{\mathbb{R}^n} f(x)\, \underbrace{e^{-\frac{1}{2}(x-a, M(x-a))}}_{e^{-\frac{1}{2}(x,Mx)}\underbrace{e^{-\frac{1}{2}(a,Ma)+(x,Ma)}}_{Z}}\, dx$$

$$= \mathbb{E}(Z f(X)).$$

Thus if $\mathbb{P}$ denotes the distribution of X then the measure $\mathbb{Q}$ with

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = Z, \quad Z \text{ as above,}$$

is that of a $\mathcal{N}(a,C)$ Gaussian vector.

Example. Let B be Brownian motion with $B_0 = 0$. Fix finitely many times $0 = t_0 < t_1 < \dots < t_n$. Then $(B_{t_i})_{i=0}^{n}$ is a centred Gaussian vector with

$$\mathbb{E}(f(B_{t_i})) = \text{const.} \int_{\mathbb{R}^n} f(x)\, e^{-\frac{1}{2}\sum_{i=1}^{n}\frac{(x_i - x_{i-1})^2}{t_i - t_{i-1}}}\, dx_1 \dots dx_n$$

Let $h : \mathbb{R}_+ \to \mathbb{R}$ be a deterministic function. Then

$$\mathbb{E}(f(B+h)) = \mathbb{E}(Z f(B)),$$

$$Z = \exp\left(-\frac{1}{2}\sum_{i=1}^{n}\frac{(h_{t_i}-h_{t_{i-1}})^2}{t_i-t_{i-1}} + \sum_{i=1}^{n}\frac{(h_{t_i}-h_{t_{i-1}})(B_{t_i}-B_{t_{i-1}})}{t_i-t_{i-1}}\right).$$

for f such that $f(B)$ depends only on $(B_{t_i})_{i=0}^{n}$.

Defn. Let M be a continuous local martingale. Then the stochastic exponential (or Doléans-Dade exponential) of M is

$$\mathcal{E}(M)_t = e^{M_t - \frac{1}{2}\langle M\rangle_t}$$

Prop. Let M be a continuous local martingale with $M_0 = 0$. Then $Z = \mathcal{E}(M)$ satisfies

$$dZ_t = Z_t\, dM_t \quad , \quad \text{i.e.,} \quad Z_t = 1 + \int_0^t Z_s \, dM_s ,$$

In particular, $\mathcal{E}(M)$ is a continuous local martingale. Moreover, if $\langle M\rangle$ is uniformly bounded then $\mathcal{E}(M)$ is a UI martingale. (More general criterion is Novikov's condition.)

Proof. By Itô's formula applied to the semimartingale $X = M - \frac{1}{2}\langle M\rangle$ and $f(x) = e^x$,

$$dZ_t = Z_t\left(dM_t - \frac{1}{2}d\langle M\rangle_t + \frac{1}{2}d\langle M\rangle_t\right) = Z_t\, dM_t .$$

Since M is a continuous local martingale, so is $Z \cdot M$ and thus Z is a continuous local martingale.

Now suppose that $\langle M\rangle_\infty \le b < \infty$. Then

$$\mathbb{P}\left(\sup_{t\ge 0} M_t \ge a\right) = \mathbb{P}\left(\sup_{t\ge 0} M_t \ge a,\ \langle M\rangle_\infty \le b\right) \le e^{-\frac{a^2}{2b}}$$

Example sheet

$$\Rightarrow \mathbb{E}\left(\exp\left(\sup M_t\right)\right) = \int_0^\infty \mathbb{P}\left(\exp\left(\sup M_t\right) \geq \lambda\right) d\lambda$$

$$= \int_0^\infty \mathbb{P}\left(\sup M_t \geq \log \lambda\right) d\lambda$$

$$\leq 1 + \int_1^\infty e^{-\frac{(\log \lambda)^2}{2b}} d\lambda < \infty$$

Since $\sup_{t\geq 0} \mathcal{E}(M)_t \leq \exp(\sup M_t)$ using that $\langle M \rangle_t \geq 0$, it follows that $\mathcal{E}(M)$ is UI.

<u>Thm</u> (Girsanov). Let M be a continuous local martingale with $M_0 = 0$. Suppose that $\mathcal{E}(M)$ is a UI martingale. Define a probability measure $\mathbb{Q}$ by

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \mathcal{E}(M)_\infty .$$

Let X be a continuous local martingale w.r.t. $\mathbb{P}$. Then $X - \langle X, M\rangle$ is a continuous local martingale w.r.t. $\mathbb{Q}$.

<u>Proof</u>. Let

$$T_n = \inf\{t \geq 0 : |X_t - \langle X, M\rangle_t| \geq n\}.$$

Then T_n is a stopping time and $\mathbb{P}(T_n \uparrow \infty) = 1$ by continuity of $X - \langle X, M\rangle$. Since $\mathbb{Q}$ is absolutely continuous w.r.t. $\mathbb{P}$ also $\mathbb{Q}(T_n \uparrow \infty) = 1$.

Thus it suffices to show that $X^{T_n} - \langle X^{T_n}, M\rangle$ is a continuous local martingale w.r.t. $\mathbb{Q}$ for every n.

Let $Y = X^{T_n} - \langle X^{T_n}, M\rangle$

$Z = \mathcal{E}(M)_\infty$.

Claim: ZY is a cont. local martingale w.r.t. $\mathbb{P}$.

$$\begin{aligned} d(ZY) &= Y_t\, dZ_t + Z_t\, dY_t + d\langle Z, Y\rangle_t \\ &= \left(X_t^{T_n} - \langle X^{T_n}, M\rangle_t\right)(Z_t\, dM_t) + Z_t\left(dX_t^{T_n} - \cancel{d\langle X^{T_n}, M\rangle_t}\right) \\ &\qquad + \cancel{Z_t\, d\langle M, X^{T_n}\rangle_t} \end{aligned}$$

All terms on the RHS are stochastic differentials w.r.t. local martingales. Thus ZY is a local martingale.

Claim: ZY is UI

Since Z is a UI martingale, $\{Z_T : T \text{ is a stopping time}\}$ is UI. ($\to$ p.18)

Since Y is bounded, $\{Z_T Y_T : T \text{ is a stopping time}\}$ is also UI.

$\Rightarrow$ ZY is a martingale w.r.t. $\mathbb{P}$. (again p.18)

$$\begin{aligned} \Rightarrow \mathbb{E}^{\mathbb{Q}}\left(Y_t - Y_s \mid \mathcal{F}_s\right) &= \mathbb{E}^{\mathbb{P}}\left(Z_\infty Y_t - Z_\infty Y_s \mid \mathcal{F}_s\right) \\ &= \mathbb{E}^{\mathbb{P}}\left(Z_t Y_t - Z_s Y_s \mid \mathcal{F}_s\right) = 0. \end{aligned}$$

Rk. The quadratic variation does not change since

$$\langle Y \rangle_t = \langle X \rangle_t = \lim_{n\to\infty} \sum_{i=1}^{\lfloor 2^n t \rfloor} (X_{i2^{-n}} - X_{(i-1)2^{-n}})^2 \quad \text{a.s. along a subsequence.}$$

Cor. Let X be a standard Brownian motion under $\mathbb{P}$, and let M be a continuous local martingale with $M_0 = 0$ s.t. $\mathcal{E}(M)$ is UI. Then $B = X - \langle X, M \rangle$ is a Brownian motion under $\mathbb{Q}$, where $d\mathbb{Q}/d\mathbb{P} = \mathcal{E}(M)_\infty$.

Proof. By Girsanov's Theorem, B is a continuous local martingale. Moreover,

$$\langle B \rangle_t = \langle X \rangle_t = t.$$

By Lévy's characterisation, thus B is a Brownian motion.

Example. Let $b : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ be bounded and assume $b(t,x) \le g(t)$ for some $g : \mathbb{R}_+ \to \mathbb{R}$ with $\int_0^\infty g(t)^2 \, dt < \infty$. Consider the SDE

$$dX_t = b(t, X_t)\, dt + dB_t$$

We can construct a solution as follows. Let X be a standard Brownian motion under $\mathbb{P}$. Set

$$M_t = \int_0^t b(s, X_s)\, dX_s.$$

Then

$$X_t - \langle X, M\rangle_t = X_t - \int_0^t b(s, X_s)\, d\langle X\rangle_s = X_t - \int_0^t b(s, X_s)\, ds$$

is a standard Brownian motion under the measure $\mathbb{Q}$ given by $d\mathbb{Q}/d\mathbb{P} = \mathcal{E}(M)_\infty$ provided that $\mathcal{E}(M)$ is a UI martingale. This is the case since $\langle M\rangle_\infty = \int_0^\infty b(s, X_s)^2\, ds \leq \int_0^\infty g(s)^2 ds < \infty$. Thus X solves the SDE.

<u>Thm</u>. Let M be a continuous local martingale s.t. $M_0 = 0$. Then

$$\mathbb{E}\left(e^{\frac{1}{2}\langle M\rangle_\infty}\right) < \infty \quad \text{(Novikov's condition)}$$

implies

M is a UI martingale and $\mathbb{E}\left(e^{\frac{1}{2}M_\infty}\right) < \infty$ (Kazamaki's condition)

which implies that

$\mathcal{E}(M)$ is a UI martingale.

4.3. The Cameron-Martin formula

Defn. The Wiener space $(W, \mathcal{W}, P)$ is given by $W = C(\mathbb{R}_+, \mathbb{R})$, $\mathcal{W} = \sigma(X_t : t \geq 0)$ where $X_t : W \to \mathbb{R}$, $X_t(w) = w(t)$, and P is the unique probababilility measure on $(W, \mathcal{W})$ s.t. (X_t) is a standard Brownian motion with $X_0 = 0$.

Defn. The Cameron-Martin space is

$$\mathcal{H} = \{ h \in W : h(t) = \int_0^t g(s)\, ds \text{ for some } g \in L^2(\mathbb{R}_+) \}.$$

For $h \in \mathcal{H}$, the function $\dot{h} = g$ is the weak derivative of h.

Exercise. $\mathcal{H}$ is a Hilbert space with inner product

$$(h, f)_{\mathcal{H}} = \int_0^\infty \dot{h}(s) \dot{f}(s)\, ds.$$

The dual space of $\mathcal{H}$ can be identified with

$$\mathcal{H}^* = \{ \mu \in M(\mathbb{R}_+) : \int_0^\infty (s \wedge t)\, \mu(ds)\, \mu(dt) = (\mu, \mu)_{\mathcal{H}^*} < \infty,\ \mu(\{0\}) = 0 \},$$

in the sense that for any bounded linear $\ell : \mathcal{H} \to \mathbb{R}$ there exists $\mu \in \mathcal{H}^*$ s.t. $\ell(h) = \int h(t)\, \mu(dt)$ and vice versa.

Rk. One would like to think of Brownian motion as the standard Gaussian measure on $\mathcal{H}$. Unfortunately, this measure does not exist. The next theorem shows it is not far.

Thm. (Cameron-Martin). Let $h \in \mathcal{H}$ and define P^h by $P^h(A) = P(\{w \in W : w+h \in A\})$ for $A \in \mathcal{W}$. The probability measure P^h on $(W, \mathcal{W})$ is absolutely continuous w.r.t. the Wiener measure and

$$\frac{dP^h}{dP} = \exp\left(\int_0^\infty \dot{h}(s)\, dX_s - \frac{1}{2}\int_0^\infty \dot{h}(s)^2\, ds\right)$$

stochastic integral, Wiener integral (Section 1.)

Proof. Let

$$M_t = \int_0^t \dot{h}(s)\, dX_s.$$

Then M is a continuous martingale w.r.t. the filtration $(\mathcal{W}_t)$, $\mathcal{W}_t = \sigma(X_s : s \le t)$.

$$\langle M \rangle_\infty = \int_0^\infty \dot{h}(s)^2\, ds = \|h\|_{\mathcal{H}}^2 < \infty.$$

$\Rightarrow \mathcal{E}(M)$ is a UI martingale.

Define Q by $dQ/dP = \mathcal{E}(M)_\infty$. Then (corollary in previous section) $Y = X - \langle X, M \rangle$ is a Q-Brownian motion since X is a P-Brownian motion.

$$\langle X, M \rangle_t = \int_0^t \dot{h}(s)\, ds = h(t).$$

$\Rightarrow Y(w) = w - h.$

$$\Rightarrow P(\{w : w+h \in A\}) = Q(\{w : \underbrace{Y(w)+h}_{w} \in A\}) = Q(A).$$

5. Stochastic Differential Equations

5.1. Definitions

In Section 1, we motivated the SDE $\dot{x}(t) = F(x(t)) + \underset{\substack{\uparrow \\ \text{white noise}}}{\eta(t)}$.

It can be interpreted as $dX_t = F(X_t)\,dt + dB_t$

$$\Leftrightarrow X_t - X_0 = \int_0^t F(X_s)\,ds + B_t.$$

<u>Defn</u>. Let $d, m \in \mathbb{N}$, $b : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$, $\sigma : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^{d \times m}$ be locally bounded (measurable). A solution to the stochastic differential equation (SDE)

$$E(\sigma, b) \qquad dX_t = b(t, X_t)\,dt + \sigma(t, X_t)\,dB_t$$

consists of

- a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ obeying the usual conditions
- an m-dimensional Brownian motion B with $B_0 = 0$.
- an $(\mathcal{F}_t)$-adapted continuous process X with values in $\mathbb{R}^d$ s.t.

$$X_t = X_0 + \underbrace{\int_0^t \sigma(s, X_s)\,dB_s}_{\sum_{j=1}^m \sigma_{ij}(s, X_s)\,dB_s^j} + \int_0^t b(s, X_s)\,ds$$

If $X_0 = x \in \mathbb{R}^d$ then X is a solution to $E_x(\sigma, b)$. It is a <u>strong solution</u> if it is adapted to the canonical filtration of B.

Defn. For the SDE $E(\sigma, b)$, there is

- *weak existence* if for every $x \in \mathbb{R}^d$ there exists a solution to $E_x(\sigma,b)$;
- *uniqueness in law* if, for every $x \in \mathbb{R}^d$, all solutions to $E_x(\sigma, b)$ have the same distribution;
- *pathwise uniqueness* if, when $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ and B are fixed, any two solutions X, X' with $X_0 = X_0'$ are indistinguishable.

Example. (Tanaka). The SDE

$$dX_t = \operatorname{sign}(X_t)\, dB_t, \quad X_0 = x \qquad \text{where } \operatorname{sign}(x) = \begin{cases} +1 & \text{if } x>0 \\ -1 & \text{if } x \le 0 \end{cases} \qquad (*)$$

has a weak solution which is unique in law, but not pathwise uniqueness.

Indeed, let X be a one-dimensional BM with $X_0 = x$. Set

$$B_t = \int_0^t \operatorname{sign}(X_s)\, dX_s,$$

which is well-defined since $\operatorname{sign}(X)$ is previsible.

$$\Rightarrow x + \int_0^t \operatorname{sign}(X_s)\, dB_s = x + \int_0^t \underbrace{\operatorname{sign}(X_s)^2}_{=1}\, dX_s = x + X_t - X_0 = X_t$$

i.e.

$$dX_t = \operatorname{sign}(X_t)\, dB_t, \quad X_0 = x.$$

Moreover, B is a Brownian motion since it is a continuous martingale and

$$\langle B, B \rangle_t = \int_0^t d\langle X, X \rangle_s = t.$$

Thus (*) has weak existence.

Any solution is a Brownian motion by the previous argument, so (*) has uniqueness in law.

Claim: if $x=0$ and X is a solution to (*) then $-X$ is also a solution with the same Brownian motion ($\Rightarrow$ pathwise uniqueness fails).

$$-X_t = -\int_0^t \mathrm{sign}(X_s)\, dB_s = \int_0^t \mathrm{sign}(-X_s)\, dB_s + \underbrace{2\int_0^t 1_{X_s=0}\, dB_s}_{N_t}$$

where N is a continuous local martingale and

$$\langle N, N\rangle_t = 4\int_0^t 1_{X_s=0}\, ds \underset{\uparrow}{=} 0$$

the zero set of BM has Lebesgue measure 0

$\Rightarrow N = 0$

$\Rightarrow -X$ also solves (*).

Rk. X is not a strong solution.

<u>Thm</u> (Yamada-Watanabe). Assume weak existence and pathwise uniqueness hold. Then:

- uniqueness in law holds
- for any $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ and B, and any $x \in \mathbb{R}^d$, there is a unique strong solution to $E_x(\sigma, b)$.

5.2. Lipschitz coefficients

Defn. The coefficients $b: \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma: \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^{d\times m}$ are Lipschitz in x if there is $K>0$ s.t. for all $t \geq 0$, $x,y \in \mathbb{R}^d$,

$$|b(t,x) - b(t,y)| \leq K|x-y|$$
$$|\sigma(t,x) - \sigma(t,y)| \leq K|x-y|.$$

Here $|\cdot|$ is any norm on $\mathbb{R}^d$ respectively $\mathbb{R}^{d\times m}$.

Thm. Assume b and σ are Lipschitz in x. Then there is pathwise uniqueness for $E(\sigma,b)$ and for every $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ satisfying the usual conditions ang every $(\mathcal{F}_t)$-Brownian motion B, for every $x \in \mathbb{R}$, there exists a unique strong solution to $E_x(\sigma, b)$.

Proof. To simplify notation, assume $d=m=1$.

Pathwise uniqueness. Let X, X' be two solutions with $X_0 = X_0'$. Set

$$S = \inf\{t \geq 0 : |X_t| \geq n \text{ or } |X_t'| \geq n\}.$$

$$\Rightarrow \quad X_{t\wedge S} = X_0 + \int_0^{t\wedge S} \sigma(s, X_s)\, dB_s + \int_0^{t\wedge S} b(s, X_s)\, ds$$

$$X'_{t\wedge S} = X_0' + \dots$$

Fix $T>0$. Then for $t \in [0,T]$,

$$\mathbb{E}\left((X_{t\wedge S} - X'_{t\wedge S})^2\right) \leq 2\, \mathbb{E}\left(\left|\int_0^{t\wedge S} (\sigma(s,X_s) - \sigma(s,X_s'))\, dB_s\right|^2\right)$$
$$+ 2\,\mathbb{E}\left(\left(\int_0^{t\wedge S} |b(s,X_s) - b(s,X_s')| ds\right)^2\right)$$

$$\Rightarrow \mathbb{E}\left((X_{t\wedge S}-X'_{t\wedge S})^2\right) \le 2\,\mathbb{E}\left(\int_0^{t\wedge S} (\sigma(s,X_s)-\sigma(s,X'_s))^2\,ds\right. +$$

$$\left. 2T\,\mathbb{E}\left(\int_0^{t\wedge S} (b(s,X_s)-b(s,X'_s))^2\,ds\right)\right)$$

$$\le 2K^2(1+T)\,\mathbb{E}\left(\int_0^{t\wedge S} |X_s-X'_s|^2\,ds\right)$$

$$\le 2K^2(1+T)\,\mathbb{E}\left(\int_0^{t} |X_{s\wedge S}-X'_{s\wedge S}|^2\,ds\right).$$

Thus $h(t)=\mathbb{E}\left(|X_{t\wedge S}-X'_{t\wedge S}|^2\right)$ satisfies $h(t)\le 4n^2$

$$h(t) \le \underbrace{2K^2(1+T)}_{\text{constant}} \int_0^t h(s)\,ds \quad \text{for } t\le T.$$

Grönwall's Lemma: if $h(t)\le a+b\int_0^t h(s)\,ds$, $h\ge 0$ is bounded on $[0,T]$,

$\Rightarrow h(t)\le a e^{bt}$ for $t\in[0,T]$.

Thus

$$\mathbb{E}\left(|X_{t\wedge S}-X'_{t\wedge S}|^2\right)=0 \text{ for } t\in[0,T].$$

Take $n\to\infty$, $T\to\infty$,

$X_t = X'_t$ for all $t\ge 0$, a.s.

Existence of a strong solution. Fix $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ and B.

Define

$$F(X)_t = X_0 + \int_0^t \sigma(s, X_s)\, dB_s + \int_0^t b(s, X_s)\, ds.$$

Then X is a strong solution to $E(\sigma, b)$ if $F(X) = X$.

To find such a fixed point, we use the Picard iteration method. Fix $T>0$. For X continuous, adapted, set

$$|||X|||_T = \mathbb{E}\Big(\sup_{t\in[0,T]} |X_t|^2 \Big)^{1/2}.$$

Then $\mathcal{B} = \{ X : \Omega \times [0,T] \to \mathbb{R} : |||X|||_T < \infty \}$ is a Banach space.

Claim: $|||F(X) - F(Y)|||_T^2 \leq (2T+8) K^2 \int_0^T |||X-Y|||_t^2 \, dt$

$$|||F(X) - F(Y)|||^2 \leq \underbrace{2\,\mathbb{E}\Big(\sup_{t\leq T} \Big| \int_0^t (b(s,X_s) - b(s,Y_s))\, ds \Big|^2 \Big)}_{(A)} + \underbrace{2\,\mathbb{E}\Big(\sup_{t\leq T} \Big| \int_0^t (\sigma(s,X_s) - \sigma(s,Y_s))\, dB_s \Big|^2 \Big)}_{(B)}$$

where

$$(A) \overset{(CS)}{\leq} T\, \mathbb{E}\Big(\sup_{t\leq T} \int_0^t |b(s,X_s) - b(s,Y_s)|^2 ds \Big)$$

$$\leq T K^2 \int_0^T |||X-Y|||_t^2 \, dt$$

$$(B) \leq 4\, \mathbb{E}\Big(\int_0^T |\sigma(s,X_s) - \sigma(s,Y_s)|^2 ds \Big) \leq 4K^2 \int_0^T |||X-Y|||_t^2 \, dt$$

Doob's inequality:

$$\mathbb{E}\Big(\sup_{t\leq T} |M_t|^2 \Big) \leq 4\, \mathbb{E}(\langle M \rangle_t)$$

Claim: $|||F(0)|||_T < \infty$

$$F(0) = X_0 + \int_0^t b(s,0)\,ds + \int_0^t \sigma(s,0)\,dB_s$$

$$\Rightarrow |||F|||_T \le |X_0| + \underbrace{|||\int_0^t b(s,0)\,ds|||}_{\overset{CS}{\le} T\int_0^T b(s,0)^2\,ds} + \underbrace{|||\int_0^t \sigma(s,0)\,dB_s|||}_{\overset{Doob}{\le} 2\,\mathbb{E}\left(\left|\int_0^T \sigma(s,0)\,dB_s\right|^2\right) \overset{Itô}{=} 2\int_0^T \sigma(s,0)^2\,ds.} < \infty.$$

(Thus F maps B to itself and is a contraction if $(2T+8)K^2T<1$.

$\Rightarrow$ F has a unique fixed point for such T.)

To define a solution to $E_x(\sigma,b)$ for all t, let $X^0_t = x$ $\forall t$.

Set $X^{i+1} = F(X^i)$.

$$\Rightarrow |||X^{i+1} - X^i|||_T^2 \le C\int_0^T |||X^i - X^{i-1}|||_t^2\,dt$$

$$\le C^2 \int_0^T\int_0^t |||X^{i-1} - X^{i-2}|||_s^2\,ds\,dt$$

$$\le |||X^1 - X^0|||_T \frac{(CT)^i}{i!}$$

$$\Rightarrow \sum_{i=1}^{\infty} |||X^i - X^{i-1}|||_T^2 < \infty \quad \forall T$$

$\Rightarrow$ X^i converges a.s. uniformly on every $[0,T]$

$\Rightarrow$ $X = F(X)$.

5.3. Examples of SDEs

5.3.1. The Ornstein-Uhlenbeck process

Let $\lambda>0$. The Ornstein-Uhlenbeck process is the solution to

$$dX_t = -\lambda X_t\,dt + dB_t$$

It is a rare example that can be solved explicitly. The solution exists by the previous theorem. By Itô's formula applied to $e^{\lambda t}X_t$:

$$d(e^{\lambda t}X_t) = e^{\lambda t}dX_t + \lambda e^{\lambda t}X_t\,dt = e^{\lambda t}dB_t$$

$$\Rightarrow e^{\lambda t}X_t - X_0 = \int_0^t e^{\lambda s}\,dB_s$$

$$\Rightarrow X_t = e^{-\lambda t}X_0 + \int_0^t \underbrace{e^{-\lambda(t-s)}}_{\text{deterministic} \to \text{Wiener integral}}\,dB_s.$$

Fact. If $X_0=x$ is fixed, (X_t) is a Gaussian process with

$$\mathbb{E}X_t = e^{-\lambda t}x, \qquad \operatorname{cov}(X_t,X_s) = \frac{1}{2\lambda}\left(e^{-\lambda|t-s|} - e^{-\lambda|t+s|}\right).$$

(Gaussian process means that $(X_{t_i})_{i=1}^n$ is jointly Gaussian for all $0\le t_1<\dots<t_n$.)

Proof. By the Itô isometry,

$$\mathbb{E}\Big(\underbrace{\Big(\int_0^t e^{-\lambda(t-u)}\,dB_u\Big)}_{X_t-\mathbb{E}X_t}\underbrace{\Big(\int_0^s e^{-\lambda(s-u)}\,dB_u\Big)}_{X_s-\mathbb{E}X_s}\Big) = e^{-\lambda(t+s)}\Big(\int_0^{t\wedge s} e^{+2\lambda u}\,du\Big) = \frac{1}{2\lambda}\left(e^{2\lambda(t\wedge s)}-1\right)e^{-\lambda(t+s)}.$$

In particular, $X_t \sim \mathcal{N}(\underbrace{e^{-\lambda t}x}_{\to 0}, \underbrace{\frac{1-e^{-2\lambda t}}{2\lambda}}_{\to \frac{1}{2\lambda}})$ for every $t>0$.

as $t \to \infty$.

Fact. If $X_0 \sim \mathcal{N}(0, \frac{1}{2\lambda})$ then (X_t) is a centred Gaussian process with stationary (only depends on differences) covariance

$$\frac{1}{2\lambda} e^{-\lambda|t-s|}.$$

5.3.2. Dyson Brownian motion

Let $\mathcal{H}_N$ be the inner product space of real symmetric $N \times N$ matrices with inner product

$$N \operatorname{Tr}(HK) \quad \text{for } X, Y \in \mathcal{H}_N.$$

Let $H^1, \dots, H^{\dim \mathcal{H}_N}$ be an orthonormal basis for $\mathcal{H}_N$.

Defn. The Gaussian Orthogonal Ensemble (GOE_N) is the standard Gaussian measure on $\mathcal{H}_N$, i.e., $H \sim GOE_N$ if

$$H = \sum_{i=1}^{\dim \mathcal{H}_N} H^i X^i$$

where $X^1, \dots, X^{\dim \mathcal{H}_N}$ are i.i.d. $\mathcal{N}(0,1)$ random variables.

Assume that each of the X^i evolves according to an independent Ornstein-Uhlenbeck process with $\lambda = \frac{1}{2}$. Then GOE_N is stationary under this process (Matrix Ornstein-Uhlenbeck process).

Thm (Dyson). The eigenvalues $\lambda_1(t) \leq \cdots \leq \lambda_N(t)$ of $H(t)$ are almost surely distinct for $t>0$ and satisfy the following autonomous system of SDEs (Dyson Brownian motion):

$$d\lambda_t^i = \left(-\frac{\lambda^i}{2} + \frac{1}{N}\sum_{j\neq i}\frac{1}{\lambda_i-\lambda_j}\right)dt + \sqrt{\frac{2}{\beta N}}\, dB^i, \qquad (DBM_\beta)$$

where $\beta=1$.

Rk. GUE_N is the standard Gaussian measure on the space of complex Hermitian matrices and GSE_N the standard Gaussian measure on the space of symplectic matrices. Their eigenvalues obey (DBM_β) with $\beta=2$ resp $\beta=4$.

Proof. Careful application of Itô's formula.

5.3.3. Geometric Brownian motion

Let $\sigma>0$ and $r\in\mathbb{R}$. Geometric Brownian motion is the solution to

$$dX_t = \sigma X_t\, dB_t + r X_t\, dt.$$

Apply Itô's formula to $\log X_t$:

$$X_t = X_0 \exp\left(\sigma B_t + \left(r-\frac{\sigma^2}{2}\right)t\right).$$

5.3.4 Bessel processes

Let $B = (B^1, \dots, B^d)$ be a d-dimensional Brownian motion. Then $X_t = |B_t|$ satisfies

$$dX_t = \frac{\nu - 1}{2X_t}\, dt + dB_t, \qquad t < \inf\{t \geq 0 : X_t = 0\}.$$

with $\nu = d$.

5.4. Representations of solutions to PDE

Prop. Assume that $b:\mathbb{R}^d\to\mathbb{R}^d$ and $\sigma:\mathbb{R}^d\to\mathbb{R}^{d\times m}$ are (measurable) locally bounded functions, and let $x\in\mathbb{R}^d$. Assume that X is a solution to $E_x(\sigma,b)$. Then for every $f\in C^1(\mathbb{R}_+)\otimes C^2(\mathbb{R}^d)$,

$$M_t^f = f(t,X_t) - f(0,X_0) - \int_0^t \left(\frac{\partial}{\partial s} + L\right) f(s,X_s)\,ds$$

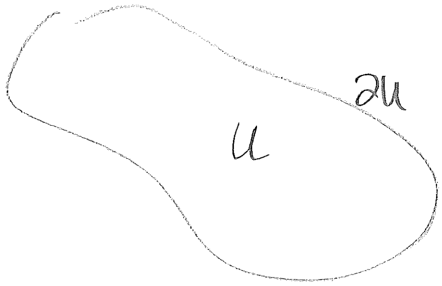
is a continuous local martingale, where

$$Lf(y) = \frac{1}{2}\sum_{i,j=1}^d a_{ij}(y)\frac{\partial^2 f}{\partial y^i \partial y^j} + \sum_{i=1}^d b_i(y)\frac{\partial f}{\partial y^i}, \quad \underbrace{a(y) = \sigma(y)\sigma(y)^T}_{\in\mathbb{R}^{d\times d}}.$$

Proof. Example Sheet.

Dirichlet-Poisson problem. Let $U\subset\mathbb{R}^d$, $U\neq\emptyset$ be bounded, open. Let $f\in C_b(U)$, $g\in C_b(\partial U)$. Find $u\in C^2(\bar{U}) = C^2(U)\cap C(\bar{U})$ s.t.

$$(\mathrm{DP})\begin{cases} -Lu(x) = f(x) & \text{for } x\in U,\\ u(x) = g(x) & \text{for } x\in\partial U.\end{cases}$$



Dirichlet problem: $f=0$

Poisson equation: $g=0$.

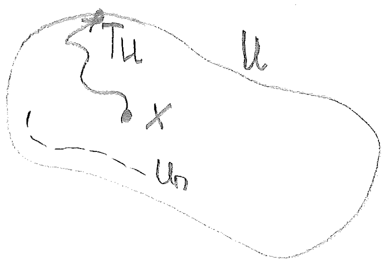
Defn. $a:\bar{U}\to\mathbb{R}^{d\times d}$ is uniformly elliptic if there is $c>0$ s.t.

$$\xi^T a(x)\xi \geq c|\xi|^2 \quad \text{for all } \xi\in\mathbb{R}^d,\ x\in\bar{U}.$$

Thm ($\to$ Gilbarg–Trudinger, Evans, ...). Assume that U has a smooth boundary (or satisfies the exterior sphere condition), that σ and b are Hölder continuous, and that $a = \sigma\sigma^T$ is uniformly elliptic. Then for every Hölder continuous $f : \bar{U} \to \mathbb{R}$ and every continuous $g : \partial U \to \mathbb{R}$, the Dirichlet–Poisson problem has a solution.

Thm. Let σ and b be bounded and measurable, let σ be uniformly elliptic, $U \subset \mathbb{R}^d$ as above. Let u be a solution to (DP). Let X be a solution to $E_x(\sigma, b)$ for some $x \in U$. Let $T_U = \inf\{t \geq 0 : X_t \notin U\}$. Then $\mathbb{E}\, T_U < \infty$ and

$$u(x) = \underbrace{\mathbb{E}_x}_{\text{emphasises that } X_0 = x}\left(g(X_{T_U}) + \int_0^{T_U} f(X_s)\, ds \right).$$



Proof. Let

$$T_n = \inf\{t \geq 0 : X_t \notin U_n\}, \quad U_n = \{x \in U : \operatorname{dist}(x, \partial U) > \tfrac{1}{n}\}.$$

There is $u_n \in C_b^2(\mathbb{R}^d)$ s.t. $u|_{U_n} = u_n|_{U_n}$. Then

$$M^n = (M^{u_n})^{T_n} = u_n(X_{t \wedge T_n}) - u_n(X_0) - \int_0^{t \wedge T_n} L u_n(X_s)\, ds$$

is a bounded local martingale, so a martingale.

$$\Rightarrow \quad \underbrace{u(x) = u_n(x)}_{\substack{\text{for } x \in U \text{ and } n \text{ large} \\ \text{enough s.t. } x \in U_n}} = \mathbb{E}\left(u(X_{t \wedge T_n}) - \int_0^{t \wedge T_n} \underbrace{Lu(X_s)}_{-f(X_s)}\, ds \right).$$

Claim: $\mathbb{E}\, T_U < \infty$

Take $f=1$ and $g=0$. Then for the corresponding solution v,

$$\mathbb{E}(t \wedge T_n) = \mathbb{E}\left(\int_0^{t\wedge T_n} Lv(X_s)\, ds \right) = v(x) - \mathbb{E}\left(v(X_{t\wedge T_n})\right).$$

Since v is bounded, by monotone convergence,

$$\mathbb{E}\, T_U < \infty.$$

Claim: $u(x) = \mathbb{E}\left(g(X_{T_U}) + \int_0^{T_U} f(X_s)\, ds\right)$.

Since $t \wedge T_n \uparrow T_U$ as $t\to\infty$, $n\to\infty$, and

$$\mathbb{E}\left(\int_0^{T_U} |f(X_s)|\, ds \right) \le \|f\|_\infty\, \mathbb{E}\, T_U < \infty$$

by DCT it follows that

$$\mathbb{E}\left(\int_0^{t\wedge T_n} f(X_s)\, ds \right) \to \mathbb{E}\left(\int_0^{T_U} f(X_s)\, ds \right).$$

Since u is continuous on $\bar{U}$, also $\mathbb{E}(u(X_{t\wedge T_n})) \to \mathbb{E}(u(X_{T_U}))$.

Cauchy Problem. For $f \in C_b^2(\mathbb{R}^d)$, find $u \in C^1(\mathbb{R}_+) \otimes C^2(\mathbb{R}^d)$ s.t.

$$(CP)\begin{cases} \frac{\partial}{\partial t} u = Lu & \text{on } (0,\infty) \times \mathbb{R}^d \\ u(0,\cdot) = f & \text{on } \mathbb{R}^d \end{cases}$$

Thm (see Gilbarg-Trudinger, Evans, ...). For every $f \in C_b^2(\mathbb{R}^d)$, there exists a solution to (CP).

Thm. Let u be a solution to (CP).
Then for any solution X to $E_x(\sigma, b)$, $x \in \mathbb{R}^d$, $0 \le s < t$,

$$\mathbb{E}(f(X_t) \mid \mathcal{F}_s) = u(t-s, X_s)$$

and in particular $u(t,x) = \mathbb{E}(f(X_t))$.

Proof. Let $g(s,x) := u(t-s, x)$. Then

$$\left(\frac{\partial}{\partial s} + L\right) g(s,x) = -\frac{\partial}{\partial t} u(t-s,x) + Lu(t-s,x) = 0$$

$\Rightarrow g(s, X_s) - g(0,x)$ is a martingale

$\Rightarrow u(t-s, X_s) = g(s, X_s) = \mathbb{E}(g(t, X_t) \mid \mathcal{F}_s) = \mathbb{E}(f(X_t) \mid \mathcal{F}_s)$.

The last theorem implies that X is a Markov process.

<u>Defn</u>. An adapted process X is a <u>Markov process</u> if

$$\mathbb{E}(\varphi(X_t) \mid \mathcal{F}_s) = \mathbb{E}(\varphi(X_t) \mid X_s)$$

for all bounded measurable φ and all $0 \le s \le t$.

<u>Cor</u>. For any bounded measurable φ, $0 \le s \le t$, there is a bounded measurable function $P_{s,t}\varphi$ s.t.

$$P_{s,t}\varphi(X_s) = \mathbb{E}(\varphi(X_t) \mid \mathcal{F}_s).$$

<u>Defn</u>. $P_{s,t}$ is called the transition operator for the Markov process. The Markov process is time-homogeneous if $P_{s,t} = P_{0,t-s}$.

<u>Thm</u> (Feynman–Kac formula). Let $f \in C_b^2(\mathbb{R}^d)$, $V \in C_b(\mathbb{R}^d)$, and suppose that $u: \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$ satisfies

$$\begin{cases} \dfrac{\partial u}{\partial t} = Lu + \underset{\nwarrow \text{ multiplication}}{Vu} \quad \text{on } \mathbb{R}_+ \times \mathbb{R}^d \\ u(0,\cdot) = f \end{cases}$$

Then, for all $t>0$, $x \in \mathbb{R}^d$, and X a solution to $E_x(\sigma, b)$,

$$u(t,x) = \mathbb{E}_x\left(f(X_t) \exp\left(\int_0^t V(X_s)\, ds\right)\right).$$

<u>Proof.</u> Let

$$E_t = \exp\left(\int_0^t V(X_s)\, ds\right).$$

For $s<t$, set $M_s = u(t-s, X_s)\, E_s$.

$$\Rightarrow dM_s = d\big(u(t-s, X_s)\big) E_s + u(t-s, X_s)\, dE_s$$

$$= \Bigg(\underbrace{\sum_{i=1}^d \frac{\partial u}{\partial x_i}(t-s, X_s)\, dX_s}_{\text{loc. martingale part.}} + \underbrace{\left(L - \frac{\partial}{\partial s}\right) u(t-s, X_s) + V(X_s)\, u(t-s, X_s)}_{0}\Bigg) E_s$$

$\Rightarrow (M_s)$ is a cont. loc. mart. on $[0,t]$

Since $(M_s)_{s \le t}$ is bounded, thus (M_s) is a martingale.

$$\Rightarrow u(t,x) = M_0 = \mathbb{E}_x M_t = \mathbb{E}_x\big(f(B_t) E_t\big).$$

6. Bonus material

6.1. Martingale Problem

As before, let $\sigma : \mathbb{R}^d \to \mathbb{R}^{d\times d}$, $b : \mathbb{R}^d \to \mathbb{R}^d$, $a = \sigma\sigma^T$ be measurable coefficients, except that now $d=m$.

Defn. A filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ together with an adapted continuous process (X_t) with values in $\mathbb{R}^d$ solves the martingale problem $M(\sigma, b)$ if

$$Y_t^i = X_t^i - \int_0^t b_i(X_s)\, ds$$

$$Y_t^i Y_t^j - \int_0^t a_{ij}(X_s)\, ds, \quad (\text{i.e., } d\langle Y^i, Y^j\rangle_t = a_{ij}(X_t)\,dt)$$

are local martingales. The martingale problem is well-posed if the possible distribution of X is unique.

Exercise. Assume that X solves $E(\sigma, b)$, or more generally, that, for every $f \in C_b^2(\mathbb{R}^d)$, $M_t^f = f(X_t) - f(X_0) - \int_0^t Lf(X_s)\, ds$ is a local martingale. Then X solves $M(\sigma, b)$.

Theorem Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$, X be a solution to $M(\sigma, b)$. Then there exists an $(\mathcal{F}_t)$-Brownian motion B defined on an enlarged probability space s.t. X solves $E(\sigma, b)$ w.r.t. this Brownian motion. (→ Example Sheet 3 under stronger assumptions.)

6.2. Convergence of Markov Chains

For $\varepsilon>0$, let $(Y_n^\varepsilon)_{n\in\mathbb{N}}$ be a Markov chain with transition probability

$$\Pi_\varepsilon(x,A) = \mathbb{P}(Y_{n+1}^\varepsilon \in A \mid Y_n = x).$$

For $t\in[0,1]$, define $(X_t^\varepsilon)_t$ to be the linear interpolation of

$$X_t^\varepsilon = Y_{t/\varepsilon}^\varepsilon \quad \text{for } t\in\varepsilon\mathbb{N}.$$

In particular, $(X_t^\varepsilon)_t$ is continuous in t.

For $\varepsilon>0$, define the rescaled transition kernel

$$K_\varepsilon(x,A) = \frac{1}{\varepsilon}\Pi_\varepsilon(x,A).$$

Define

$$a_{ij}^\varepsilon(x) = \int_{y\in B_1(x)} (y^i-x^i)(y^j-x^j)\,K_\varepsilon(x,dy)$$

$$b_i^\varepsilon(x) = \int_{y\in B_1(x)} (y^i-x^i)\,K_\varepsilon(x,dy)$$

<u>Thm</u>. Let $\sigma:\mathbb{R}^d\to\mathbb{R}^{d\times d}$ and $b:\mathbb{R}^d\to\mathbb{R}^d$ be continuous coefficients s.t. $M(\sigma,b)$ is well-posed. Assume that

(i) $\lim_{\varepsilon\to0}\sup_{|x|\le R} |a_{ij}^\varepsilon(x)-a_{ij}(x)|=0 \quad \forall R>0$

(ii) $\lim_{\varepsilon\to0}\sup_{|x|\le R} |b_i^\varepsilon(x)-b_i(x)|=0 \quad \forall R>0$

(iii) $\lim_{\varepsilon\to0}\sup_{|x|\le R} K_\varepsilon(x,B_\delta(x)^c)=0 \quad \forall R>0,\delta>0.$

Then, if $X_0^\varepsilon = x_0^\varepsilon \to x_0 \in \mathbb{R}^d$ as $\varepsilon \to 0$, the process $(X_t^\varepsilon)_t$ converges weakly to (X_t) in $C([0,1], \mathbb{R}^d)$, where X is the solution to $M(a,b)$.

Recall and define

$$Lf(x) = \frac{1}{2} \sum_{i,j} a_{ij}(x) \frac{\partial^2 f}{\partial x^i \partial x^j}(x) + \sum_i b_i(x) \frac{\partial f}{\partial x^i}(x)$$

$$L^\varepsilon f(x) = \int K_\varepsilon(x, dy) \, (f(y) - f(x))$$

<u>Lemma</u>. Assumptions (i), (ii), (iii) are equivalent to

$$L^\varepsilon f(x) \xrightarrow{\varepsilon \to 0} Lf(x) \qquad \forall f \in C_c^\infty(\mathbb{R}^d)$$

uniformly on compact subsets of $\mathbb{R}^d$.

<u>Proof</u>. Assume (i), (ii), (iii). We show $L^\varepsilon f \to Lf$. Fix $f \in C_c^\infty$.

First, define $\tilde{L}^\varepsilon$ like L but with a replaced by a^ε and with b replaced by b^ε. Then (i), (ii) imply

$$\tilde{L}^\varepsilon f \to Lf \quad \text{uniformly on compact sets.}$$

Claim: $\tilde{L}^\varepsilon f - L^\varepsilon f \to 0$ uniformly on compact sets.

By Taylor's theorem, there is a constant C_f s.t.

$$\left| f(y) - f(x) - \sum_{i=1}^d (y_i - x_i) \frac{\partial f}{\partial x_i}(x) - \frac{1}{2} \sum_{i,j=1}^d (y_i - x_i)(y_j - x_j) \frac{\partial^2 f}{\partial x^i \partial x^j}(x) \right| \le C_f |y-x|^3.$$

Thus

$$\left| L^\varepsilon f(x) - \tilde{L}^\varepsilon f(x) \right| \le \underbrace{C_f \int_{B_\delta(x)} |y-x|^3 K_\varepsilon(x, dy)}_{\le 2\delta \sum_i \sup_{|x| \le R} |a_{ii}(x)|} + \underbrace{\int_{B_\delta(x)^c} |f(y) - f(x)| \, K_\varepsilon(x, dy)}_{\substack{\le 2\|f\|_\infty K_\varepsilon(x, B_\delta(x)^c) \text{ by (iii)} \\ \to 0 \ (\varepsilon \to 0)}}$$

Taking first δ small then $\varepsilon \to 0$,

$$L^\varepsilon f - \tilde{L}^\varepsilon f \to 0 \quad \text{uniformly on compact sets}$$

The direction $L^\varepsilon f \to Lf$ $\forall f \in C_c^\infty$ implies (i), (ii), (iii) is left as exercise.

Lemma. Let P^ε denote the law of X^ε, i.e., P^ε is a probability measure on $C([0,1], \mathbb{R}^d)$. Then the family $(P_\varepsilon)_\varepsilon$ is tight, i.e., there is a subsequence ε_k s.t. $P_{\varepsilon_k} \to P$ in the weak-$*$ topology, for some subsequence (ε_k)

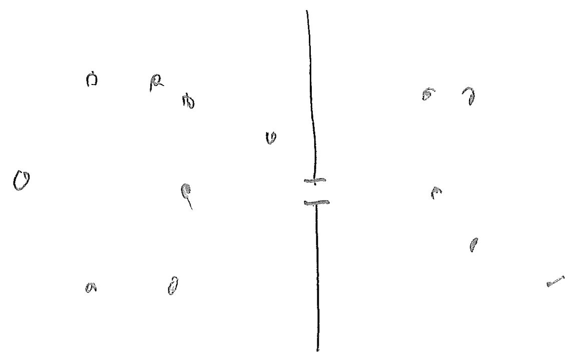
Proof. See N. Berestycki's notes.

Proof of Theorem. Let (ε_k) be s.t. $P^{\varepsilon_k} \to P$ weakly. It suffices to show that P solves $M(a,b)$.

Note that:

- $$f(X_{\varepsilon_k n}) - \sum_{m=0}^{k-1} \int \left(f(y) - f(X_{\varepsilon_k m})\right) P^{\varepsilon_k}(X_{\varepsilon_k m}, dy)$$
  is a (discrete) $(\mathcal{F}_{\varepsilon_k m})_m$-martingale
- $$f\left(X_{\varepsilon_k \lfloor \frac{t}{\varepsilon_k} \rfloor}\right) - f\left(X_{\varepsilon_k \lfloor \frac{s}{\varepsilon_k} \rfloor}\right) - \sum_{m=\lfloor \frac{s}{\varepsilon_k} \rfloor}^{\lfloor \frac{t}{\varepsilon_k} \rfloor} \int \left(f(y) - f(X_{m\varepsilon_k})\right) P^{\varepsilon_k}(X_{m\varepsilon_k}, dy)$$
  $$\to f(X_t) - f(X_s) - \int_s^t Lf(X_u)\, du \quad \text{a.s.}$$

Example



Consider $2m$ particles. At each time step, a uniformly chosen particle switches sides.

Let $N_n^{(m)}$ be the number of particles on the left.

Let $X_t^{(m)} = \dfrac{N^{(m)}_{\lfloor mt \rfloor} - m}{\sqrt{m}}$.

Thm. Assume $X_0^{(m)} \to \alpha \in \mathbb{R}$. Then $(X_t^{(m)})_{t \le T}$ converges weakly to the the unique solution to

$$\begin{cases} dX_t = -X_t\,dt + dB_t \\ X_0 = \alpha \end{cases}$$

Indeed, $X_t^{(m)} \in \{\frac{k}{\sqrt{m}}, -m \le k \le m\}$ and

$$\Pi^{(m)}\left(x, x + \tfrac{1}{\sqrt{m}}\right) = \frac{1}{2} - \frac{x}{2\sqrt{m}}, \quad \Pi^{(m)}\left(x, x - \tfrac{1}{\sqrt{m}}\right) = \frac{1}{2} + \frac{x}{2\sqrt{m}}.$$

This gives (with $\varepsilon = 1/m$)

$$a^{(m)}(x) = m \int \left(\frac{1}{\sqrt{m}}\right)^2 \Pi^{(m)}(x, dy) = 1, \quad b^{(m)}(x) = -x$$

and the claim follows from the general theorem.