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**Problem 1.** Suppose that $(Z_t)_{t\geq 0}$ is a continuous local martingale which is strictly positive almost surely. Show that there is a unique continuous local martingale $M$ such that $Z = \mathcal{E}(M)$, where

$$\mathcal{E}(M)_t = \exp(M_t - \tfrac{1}{2}\langle M\rangle_t).$$

**Problem 2.** Let $M$ be a continuous local martingale with $M_0 = 0$. For any $a, b > 0$, show that

$$\mathbb{P}\left(\sup_{t\geq 0} M_t \geq a, \langle M\rangle_\infty \leq b\right) \leq \exp\left(-\frac{a^2}{2b}\right).$$

**Problem 3.** Let $B$ be a standard Brownian motion and, for $a, b > 0$, let $\tau_{a,b} = \inf\{t \geq 0 : B_t + bt = a\}$. Use Girsanov's theorem to prove that the density of $\tau_{a,b}$ is given by

$$a(2\pi t^3)^{-1/2}\exp(-(a-bt)^2/2t).$$

**Problem 4.** Suppose that $M$ is a continuous local martingale with $\langle M\rangle_t \to \infty$ almost surely as $t \to \infty$. Show that $M_t/\langle M\rangle_t \to 0$ as $t \to \infty$ and conclude that $\mathcal{E}(M)_t \to 0$ almost surely.

**Problem 5.** [Gronwall's lemma] Let $T > 0$ and let $f$ be a non-negative, bounded, measurable function on $[0, T]$. Suppose that there exist $a, b \geq 0$ such that

$$f(t) \leq a + b\int_0^t f(s)ds \quad \text{for all} \quad t \in [0, T].$$

Show that $f(t) \leq ae^{bt}$ for all $t \in [0, T]$.

**Problem 6.** Suppose that $X$ is a continuous local martingale with quadratic variation

$$\langle X\rangle_t = \int_0^t A_s ds$$

for a non-negative, previsible process $(A_t)_{t\geq 0}$. Show that there exists a Brownian motion $B$ (possibly defined on a larger probability space) such that

$$X_t = \int_0^t A_s^{1/2} dB_s.$$

**Problem 7.** Suppose that $\sigma$ and $b$ are Lipschitz. Explain why uniqueness in law holds for the SDE $dX_t = \sigma(X_t)dB_t + b(X_t)dt$.

**Problem 8.** Suppose that $\mathbb{Q} \ll \mathbb{P}$. Show that if $X_n \to X$ in probability with respect to $\mathbb{P}$, then $X_n \to X$ in probability with respect to $\mathbb{Q}$.

**Problem 9.** Suppose that $\sigma, b$ and $\sigma_n, b_n$ for $n \in \mathbb{N}$ are Lipschitz with constant $K$ uniformly in $n$. Suppose that $\sigma_n \to \sigma$ and $b_n \to b$ uniformly. Suppose that $X$ and $X^n$ are defined by

$$dX_t = \sigma(X_t)dB_t + b(X_t)dt, \quad X_0 = x \tag{1}$$

$$dX_t^n = \sigma_n(X_t^n)dB_t + b_n(X_t^n)dt, \quad X_0^n = x. \tag{2}$$

Show for each $t > 0$ that

$$\mathbb{E}\left(\sup_{0\leq s\leq t} |X_s^n - X_s|^2\right) \to 0 \quad \text{as} \quad n \to \infty.$$

**Bonus**: Suppose that $b_n, \sigma_n$ are continuous, and $b, \sigma$ are Lipschitz. Suppose that $X^n$ still satisfy (1). What happens now?

**Problem 10.** Let $b$ be bounded and $\sigma$ be bounded and continuous.

*(i)* Suppose that $X$ is a weak solution of the SDE $dX_t = b(X_t)dt + \sigma(X_t)dW_t$. Show that the process

$$f(X_t) - \int_0^t \left( b(X_s)f'(X_s) - \frac{1}{2}\sigma^2(X_s)f''(X_s) \right) ds$$

is a local martingale for all $f \in C^2$.

*(ii)* Let $X$ be a continuous, adapted process such that

$$f(X_t) - \int_0^t \left( b(X_s)f'(X_s) - \frac{1}{2}\sigma^2(X_s)f''(X_s) \right) ds$$

is a local martingale for each $f \in C^2$. Suppose that $\sigma(x) > 0$ for all $x$. Show that there exists a Brownian motion such that $dX_t = b(X_t)dt + \sigma(X_t)dW_t$. (Hint: use Problem 6.)

**Problem 11.** Let $W$ be a standard Brownian motion.

*(i)* Let $B_t = W_t - tW_1$. Show that $(B_t)_{t\in[0,1]}$ is a continuous, mean-zero Gaussian process. What is the covariance $\mathbb{E}[B_s B_t]$?

*(ii)* Is $B$ adapted to the filtration generated by $W$?

*(iii)* Let

$$dX_t = -\frac{X_t}{1-t}dt + dW_t, \quad X_0 = 0.$$

Verify that

$$X_t = (1-t)\int_0^t \frac{dW_s}{1-s} \quad \text{for} \quad 0 \le t < 1.$$

Show that $X_t \to 0$ as $t \uparrow 1$.

*(iv)* Show that $X$ is a continuous, mean-zero Gaussian process with the same covariance as $B$, i.e., $X$ is a Brownian bridge.

**Problem 12 (⋆).** Using the results of this course, give a *short* proof of the reflection principle: if $T$ is a stopping time and $B$ is a standard Brownian motion, then

$$W_t = \begin{cases} B_t & t \le T; \\ 2B_T - B_t & t > T. \end{cases}$$

is also a standard Brownian Motion.