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Problem 1. Suppose that M is a continuous local martingale with $M_0 = 0$. Show that M is an L^2-bounded martingale if and only if $\mathbb{E}(\langle M\rangle_\infty) < \infty$.

Problem 2.

(i) Suppose that M, N are independent continuous local martingales. Show that $\langle M, N\rangle = 0$. In particular, if $B^{(1)}$ and $B^{(2)}$ are the coordinates of a standard Brownian motion in $\mathbb{R}^2$, this shows that $\langle B^{(1)}, B^{(2)}\rangle_t = 0$ for all $t \geq 0$.

(ii) Let B be a standard Brownian motion in $\mathbb{R}$ and let T be a stopping time which is a.s. not constant. By considering B^T and $B - B^T$, show that the converse to the previous part is false. Hint: show that T is measurable with respect to the σ-algebras generated by both B^T and $B - B^T$.

Problem 3. (Burkholder inequality) Fix $p \geq 2$ and let M be a continuous local martingale with $M_0 = 0$. Use Itô's formula, Doob's inequality, and Hölder's inequality to show that there exists a constant $C_p > 0$ such that

$$\mathbb{E}\left(\sup_{0\leq s\leq t} |M_s|^p\right) \leq C_p \mathbb{E}\big(\langle M\rangle_t^{p/2}\big).$$

Problem 4. Suppose that $f\colon [0,\infty) \to \mathbb{R}$ is a continuous function. Show that if f has finite variation then it has zero quadratic variation. Conversely, show that if f has finite and positive quadratic variation then it must be of infinite variation.

Problem 5. Let B be a standard Brownian motion. Use Itô's formula to show that the following are martingales with respect to the filtration generated by B.

(i) $X_t = \exp(\lambda^2 t/2)\sin(\lambda B_t)$

(ii) $X_t = (B_t + t)\exp(-B_t - t/2)$

(iii) $X_t = \exp(B_t - t/2)$

Problem 6. Let $h\colon [0,\infty) \to \mathbb{R}$ be a measurable function which is square-integrable when restricted to $[0, t]$ for each $t > 0$ and let B be a standard Brownian motion. Show that the process $H_t = \int_0^t h(s)dB_s$ is Gaussian and compute its covariance. (A real-valued process (X_t) is Gaussian if for any finite family $0 \leq t_1 < t_2 < \cdots < t_n < \infty$, the random vector $(X_{t_1}, \ldots, X_{t_n})$ is Gaussian).

Problem 7. Show that convergence in $(\mathcal{M}_c^2, \|\cdot\|)$ implies ucp convergence.

Problem 8. Show that the covariation $\langle\cdot,\cdot\rangle$ is symmetric and bilinear. That is, if M_1, M_2, M_3 are continuous local martingales and $a \in \mathbb{R}$, then

$$\langle aM_1 + M_2, M_3\rangle = a\langle M_1, M_3\rangle + \langle M_2, M_3\rangle.$$

Problem 9. Let B be a standard Brownian motion and let

$$\widehat{B}_t = B_t - \int_0^t \frac{B_s}{s} ds.$$

(i) Show that $\widehat{B}$ is not a martingale in the filtration generated by B.

(ii) Show that $\widehat{B}$ is a martingale in its own filtration by showing that it is a Brownian motion. [Hint: show that $\widehat{B}$ is a continuous Gaussian process and identify its mean and covariance.]

Problem 10. Fix $d \geq 3$ and let B be a Brownian motion in $\mathbb{R}^d$ starting at $B_0 = \overline{x} = (x, 0, \ldots, 0) \in \mathbb{R}^d$ for some $x > 0$. Let $|\cdot|$ denote the Euclidean norm on $\mathbb{R}^d$. For each $a > 0$, let $\tau_a = \inf\{t > 0 : |B_t| = a\}$.

(i) Let $D = \mathbb{R}^d \setminus \{0\}$ and let $h \colon D \to \mathbb{R}$ be defined by $h(x) = |x|^{2-d}$. Show that h is harmonic on D and that $M_t = |B_t^{\tau_a}|^{2-d}$ is a local martingale for all $a \geq 0$. For which values of x is M a true martingale?

(ii) Use the previous part to show that for any $a < b$ such that $0 < a < x < b$,

$$\mathbb{P}_{\overline{x}}[\tau_a < \tau_b] = \frac{\phi(b) - \phi(x)}{\phi(b) - \phi(a)}$$

where $\phi(u) = u^{2-d}$. Conclude that if $x > a > 0$, then

$$\mathbb{P}_x[\tau_a < \infty] = (a/x)^{d-2}.$$

Problem 11.

(i) Let $f \colon \mathbb{C} \to \mathbb{C}$ be analytic and let $Z_t = X_t + iY_t$ where (X, Y) is a Brownian motion in $\mathbb{R}^2$. Use Itô's formula to show that $M = f(Z)$ is a local martingale in $\mathbb{R}^2$. Show further that M is a time-change of Brownian motion in $\mathbb{R}^2$.

(ii) Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and fix $z \in \mathbb{D}$. What is the hitting distribution for Z on ∂D in the case that $Z_0 = 0$? By applying a Möbius transformation $\mathbb{D} \to \mathbb{D}$ and using the previous part, determine the hitting distribution for Z on $\partial\mathbb{D}$.

Problem 12. **($\star$)** Let $U \subset \mathbb{R}^d$ be an open set. We say that a function $u \in L^\infty_{\text{loc}}(U)$ satisfies the *mean value property* if, whenever $S(x, r) \subset U$, we have

$$u(x) = \int_{S(x,r)} u(y) \mu_{x,r}(dy) \tag{1}$$

where we write $\mu_{x,r}$ for the uniform distribution on the sphere $S(x, r) = \partial B(x, r)$.

(i) Suppose $u \in C^2(U)$ is harmonic. Show that u satisfies (1).

(ii) Suppose, conversely, that u satisfies (1). For any compact $K \subset U$, express $u|_K$ as a convolution, and deduce that $u \in C^\infty(U)$.

(iii) Suppose u satisfies (1). Fix $x \in U$ and $r > 0$ such that $\overline{B(x, r)} \subset U$. Let B be a d-dimensional Brownian Motion started at x, and let $\tau_r = \inf\{t > 0 : |x - B_t| = r\}$. Show that

$$\forall t \geq 0, \qquad \mathbb{E}\left(\int_0^{t \wedge \tau_r} \triangle u(B_s) ds\right) = 0.$$

Deduce that u is harmonic. Hence (1) is an equivalent characterisation of harmonic functions.

Problem 13. **($\star$)** (Liouville's Theorem.) Suppose $u : \mathbb{R}^d \to \mathbb{R}$ is bounded and harmonic. Let B be a Brownian motion starting at 0.

(i) Show that $M_t = u(B_t)$ is a bounded martingale. Conclude that M_t converges, almost surely and in L^1, to a random variable M_∞.

(ii) Recall Blumenthal's $0 - 1$ law. Deduce that the *tail σ-algebra*

$$\tau = \cap_{t \geq 0} \, \sigma(B_s : s \geq t)$$

contains only events of probability 0 and 1. Deduce that M_∞ is almost surely constant.

(iii) Using the relationship between M_∞ and M_1, deduce that M_1 is almost surely constant. Conclude that u is constant.