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Problem 1. Let $f\colon [0,\infty) \to \mathbb{R}$ is absolutely continuous, in the sense that

$$f(t) = f(0) + \int_0^t f'(s)ds \quad \text{for all} \quad t \geq 0$$

for an integrable function f'. Let $v(t)$ be the total variation of f on $[0,t]$. Show that

$$v(t) = |f(0)| + \int_0^t |f'(s)|ds.$$

Problem 2. Let $f, g\colon [0,\infty) \to \mathbb{R}$ be bounded and measurable and let $a\colon [0,\infty) \to \mathbb{R}$ be continuous and of finite variation. Show that

$$f \cdot (g \cdot a) = (fg) \cdot a$$

where $\cdot$ denotes the Lebesgue-Stieltjes integral.

Problem 3. Suppose that $(f^n)_{n\geq 1}$ is a sequence of càdlàg functions such that $f^n \to f$ uniformly on $[0,t]$. Show that f is càdlàg on $[0,t]$. Let f be a right-continuous function of bounded variation. Show that v_f is right-continuous.

Problem 4. Let H be a previsible process. Let $\mathcal{F}_{t^-} = \sigma(\mathcal{F}_s : s < t)$. Show that H_t is $\mathcal{F}_{t^-}$-measurable, for any $t > 0$.

Problem 5. Let $(\mathcal{F}_t)_{t\geq 0}$ be a filtration, let T be a stopping time, and let

$$\mathcal{F}_T = \{A \in \mathcal{F} : A \cap \{T \leq t\} \in \mathcal{F}_t \quad \forall t \geq 0\}.$$

(i) Show that $\mathcal{F}_T$ is a σ-algebra.

(ii) Show that T is $\mathcal{F}_T$-measurable.

(iii) Suppose that X is a càdlàg, adapted process. Show that X_T is $\mathcal{F}_T$-measurable.

Problem 6. Let $(T_n)_{n\geq 1}$ denote a sequence of stopping time for a filtration $(\mathcal{F}_t)_{t\geq 0}$.

(i) Show that $\sup\limits_{n\geq 1} T_n$ is a stopping time for $(\mathcal{F}_t)_{t\geq 0}$.

(ii) Show that T is stopping time for the filtration $\mathcal{F}_{t^+} = \bigcap_{s>t} \mathcal{F}_s$ iff $\{T < t\} \in \mathcal{F}_{t^+}$ for all t.

(iii) Show that $\inf\limits_{n\geq 1} T_n$ is a stopping time for $(\mathcal{F}_{t^+})_{t\geq 0}$.

Problem 7. Let B be a standard Brownian motion.

(i) Let $T = \inf\{t \geq 0 : B_t = 1\}$. Show that H defined by $H_t = \mathbf{1}\{T \geq t\}$ is previsible.

(ii) Let

$$\operatorname{sgn}(x) = \begin{cases} -1 & \text{if} \quad x \leq 0 \\ 1 & \text{if} \quad x > 0. \end{cases}$$

Show that $(\operatorname{sgn}(B_t))_{t\geq 0}$ is a previsible process which is neither left nor right continuous.

Problem 8. Let N be a Poisson process of rate 1, and let $X_t = N_t - t$ for $t \geq 0$. Show that X is of finite variation. Show that both X and $X_t^2 - t$ are martingales.

Problem 9. Let T and ξ denote two independent random variables defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with

$$\mathbb{P}(T \leq t) = t \quad \text{for } t \in [0,1], \qquad \mathbb{P}(\xi = 1) = \mathbb{P}(\xi = -1) = 1/2.$$

Define $X_t = \xi \mathbf{1}_{t \geq T}$ and $\mathcal{F}_t = \sigma(X_s : s \leq t)$. Show that X is a martingale with respect to $(\mathcal{F}_t)_{t \in [0,1]}$, and that it is of finite variation. Define pathwise

$$Y_t(\omega) := \int_{(0,t]} X_s(\omega) dX_s(\omega) \qquad \text{for all } \omega \in \Omega,$$

where the right-hand side is a Lebesgue-Stieltjes integral. Show that $Y = (Y_t)_{t \in [0,1]}$ is not a martingale. Is X previsible?

Problem 10. Suppose that $X \in L^1(\Omega, \mathcal{F}, \mathbb{P})$. Show that the family

$$\mathcal{X} = \{\mathbb{E}[X \mid \mathcal{G}] : \mathcal{G} \subseteq \mathcal{F} \quad \text{is a } \sigma\text{-algebra}\} \qquad \text{is UI.}$$

Problem 11. Let X be a continuous local martingale. Show that if

$$\mathbb{E}\Big(\sup_{0 \leq s \leq t} |X_s|\Big) < \infty \qquad \forall t \geq 0$$

then X is a martingale.

Problem 12. Let B be a standard Brownian motion and fix $t \geq 0$. For $n \geq 1$, let $\Delta_n = \{0 : t_0(n) < t_1(n) < \cdots < t_{m_n}(n) = t\}$ be a partition of $[0,t]$ such that

$$\max_{1 \leq i \leq m_n} \big(t_i(n) - t_{i-1}(n)\big) \to 0 \quad \text{as} \quad n \to \infty.$$

Show that

$$\lim_{n \to \infty} \sum_{i=1}^{m_n} (B_{t_i} - B_{t_{i-1}})^2 = t$$

in L^2. Show that if the subdivision is dyadic, then the convergence is also almost sure.

Bonus: Show that the convergence is almost sure for any nested subdivision. Hint: see Question 6, `https://www.maths.cam.ac.uk/sites/www.maths.cam.ac.uk/files/pre2014/postgrad/mathiii/pastpapers/2011/PaperIII_28.pdf`.

Problem 13. (A silly martingale) Construct a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$, a L^∞-bounded martingale $(M_t)_{t=0}^{t=1}$ and a stopping time T taking values in $[0,1]$, such that

$$\mathbb{E}(M_T) \neq \mathbb{E}(M_0).$$