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Exercise 1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and $(A_n)_{n\geq 1}$ be a sequence of independent events. We denote $a_n = \mathbb{P}(A_n)$ and define $b_n = a_1 + \cdots + a_n$, $S_n = \mathbb{1}_{A_1} + \cdots + \mathbb{1}_{A_n}$. Assuming $b_n \to \infty$ as $n \to \infty$, prove that S_n/b_n converges almost surely.

Exercise 2. Let $(X_n)_{n\geq 1}$ be i.i.d. Bernoulli random variables with parameter $p \in (0,1)$, i.e. $\mathbb{P}(X_i = 1) = 1 - \mathbb{P}(X_i = 0) = p$. Let N be a Poisson random variable with parameter $\lambda > 0$, i.e. for any $k \geq 0$ we have $\mathbb{P}(N = k) = e^{-\lambda}\frac{\lambda^k}{k!}$. Assume N is independent from $(X_n)_{n\geq 1}$.

Let $P = \sum_{i=1}^{N} X_i$, $F = N - P$.

a) What is the joint distribution of (P, N)?

b) Prove that P and F are independent.

Exercise 3. Let Y be an integrable random variable on $(\Omega, \mathcal{A}, \mathbb{P})$ and $\mathcal{G}$ a sub σ-field of $\mathcal{A}$. Suppose that $\mathcal{H} \subset \mathcal{G}$ is a sub σ-field of $\mathcal{G}$. Show that $\mathbb{E}\left(\mathbb{E}(Y \mid \mathcal{G}) \mid \mathcal{H}\right) = \mathbb{E}\left(Y \mid \mathcal{H}\right)$ (almost surely).

Exercise 4. Let $X_1, \ldots, X_n$ be i.i.d. integrable random variables, and $S = \sum_{i=1}^{n} X_i$. Calculate $\mathbb{E}[S \mid X_1]$ and $\mathbb{E}[X_1 \mid S]$.

Exercise 5. For fixed $a, b > 0$, let (X, Y) be a $\mathbb{N} \times \mathbb{R}_+$-valued random variable such that

$$\mathbb{P}(X = n, Y \leq t) = b \int_0^t \frac{(ay)^n}{n!} e^{-(a+b)y} \mathrm{d}y.$$

For $h : \mathbb{R}_+ \to \mathbb{R}$ continuous and bounded, calculate $\mathbb{E}[h(Y) \mid X]$. Calculate $\mathbb{E}[\frac{Y}{X+1}]$. Calculate $\mathbb{P}(X = n \mid Y)$. Calculate $\mathbb{E}[X \mid Y]$.

Exercise 6. Let X and Y be random variables on $(\Omega, \mathcal{F}, \mathbb{P})$, and $\mathcal{G}, \mathcal{H}$ sub σ-fields of $\mathcal{F}$ such that $\sigma(\mathcal{G}, \mathcal{H}) = \mathcal{F}$. Find counterexamples to the following assertions:

(i) If $\mathbb{E}[X \mid Y] = \mathbb{E}[X]$ then X and Y are independent.

(ii) If $\mathbb{E}[X \mid \mathcal{G}] = \mathbb{E}[X \mid \mathcal{H}] = 0$ then $X = 0$.

(iii) If X and Y are independent then so are $\mathbb{E}[X \mid \mathcal{G}]$ and $\mathbb{E}[Y \mid \mathcal{G}]$.

Exercise 7. Let $(X_n)_{n\geq 1}$ be a sequence of nonnegative random variables on $(\Omega, \mathcal{A}, \mathbb{P})$, and $(\mathcal{F}_n)_{n\geq 0}$ a sequence of sub σ-fields of $\mathcal{F}$. Assume that $\mathbb{E}(X_n \mid \mathcal{F}_n)$ converges to 0 in probability.

(i) Show that X_n converges to 0 in probability.

(ii) Show that the converse statement is wrong.

Exercise 8. On the same probability space, let X, Y be positive random variables such that $\mathbb{E}[X \mid Y] = Y$ and $\mathbb{E}[Y \mid X] = X$ (almost surely). Prove that $X = Y$ almost surely.