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Exercise 1. For fixed $p, q \in [0,1]$, consider a Markov chain X with two states $\{1,2\}$, with transition matrix

$$\pi = (\pi(i,j))_{1 \le i,j \le 2} = \begin{pmatrix} 1-p & p \\ q & 1-q \end{pmatrix}$$

(i) For which p, q is the chain irreducible? Aperiodic?
(ii) What are the invariant probability measures of X?
(iii) Compute $\pi^{(n)}$, $n \ge 1$.
(iv) When X is irreducible, for this invariant probability measure μ, calculate

$$d_1(n) = \frac{1}{2}\left(|\mathbb{P}_1(X_n = 1) - \mu(1)| + |\mathbb{P}_1(X_n = 2) - \mu(2)|\right)$$

$$d_2(n) = \frac{1}{2}\left(|\mathbb{P}_2(X_n = 1) - \mu(1)| + |\mathbb{P}_2(X_n = 2) - \mu(2)|\right)$$

where $\mathbb{P}_x$ means the chain starts at x.

Exercise 2. Let $X = (X_n)_{n\ge 0}$ be a sequence of fair coin tosses (with the two possible outcomes interpreted as 0 and 1) and set $M_n = \max_{k\le n} X_k$. Show that $(M_n)_{n\ge 0}$ is a Markov chain and find the transition probabilities.

Exercise 3. (Harder) Let $S = (S_n)_{n\ge 0}$ be a simple (possibly asymmetric) random walk on $\mathbb{Z}$ with $S_0 = 0$. Show that $X_n = |S_n|$ defines a Markov chain and find its transition probabilities. Let $M_n = \max_{k\le n} S_k$ and show that $Y_n = M_n - S_n$ defines a Markov chain.

Exercise 4. Let $X = (X_n)_{n\ge 0}$ and $Y = (Y_n)_{n\ge 0}$ be Markov chains on the integers $\mathbb{Z}$. Is $Z_n = X_n + Y_n$ necessarily a Markov chain. Justify your answer.

Exercise 5. Let $Y_1, Y_2, \ldots$ be i.i.d. random variables with $\mathbb{P}[Y_1 = 1] = \mathbb{P}[Y_1 = -1] = 1/2$ and set $X_0 = 1$, $X_n = X_0 + Y_1 + \ldots Y_n$ for $n \ge 1$. Define

$$H_0 = \inf\{n \ge 0 : X_n = 0\}.$$

Find the probability generating function $\phi(s) = \mathbb{E}[s^{H_0}]$.

Suppose the common distribution of the Y_i is changed to $\mathbb{P}[Y_1 = 2] = \mathbb{P}[Y_1 = -1] = 1/2$. Show that the probability generating function ϕ now satisfies

$$s\phi^3 - 2\phi + s = 0.$$

Exercise 6. Let $\pi(x,y) = p(x-y)$ be the transition probability probability of a simple random walk on $\mathbb{Z}^d$, symmetric or not. By considering the characteristic function of p, decide in which cases the random walk is transient and recurrent.

Exercise 7. For an irreducible recurrent Markov chain, prove that for any x, y,

$$\frac{E_x[N_n(y)]}{n} \to \frac{P_x[T_y < \infty]}{E_y[T_y]} \tag{0.1}$$

where $N_n(y) = \sum_{k=1}^n 1_{X_k = y}$ is the number of visits to y up to time n.

In the recurrent cases of the previous exercise, use this to decide whether the simple random walk are null or positive recurrent.