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Exercise 1. Let $(X_n)_{n\geq 1}$ be a sequence of random variables, on the same probability space, with $\mathbb{E}(X_\ell) = \mu$ for any ℓ, and a weak correlation in the following sense: $\mathrm{Cov}(X_k, X_\ell) \leq f(|k-\ell|)$ for all indexes k, ℓ, where the sequence $(f(m))_{m\geq 0}$ converges to 0 as $m \to \infty$. Prove that $(n^{-1}\sum_{k=1}^n X_k)_{n\geq 1}$ converges to μ in L^2.

Exercise 2. A sequence of random variables $(X_i)_{i\geq 1}$ is said to be completely convergent to X if for any $\varepsilon > 0$, we have $\sum_{i\geq 1}\mathbb{P}(|X_i - X| > \varepsilon) < \infty$. Prove that complete convergence implies almost sure convergence.

Exercise 3. Let X and Y be independent Gaussian random variables with null expectation and variance 1. Show that $\frac{X+Y}{\sqrt{2}}$ and $\frac{X-Y}{\sqrt{2}}$ are also independent $\mathcal{N}(0,1)$.

Exercise 4. For any $d \geq 1$, we admit that there is only one probability measure μ on $\mathbb{S}^d$, (the d-dimensional sphere embedded in $\mathbb{R}^{d+1}$) that is uniform, in the following sense: for any isometry $A \in \mathrm{O}(d+1)$ (the orthogonal group in $\mathbb{R}^{d+1}$) and any continuous function $f : \mathbb{S}^d \to \mathbb{R}$,

$$\int_{\mathbb{S}^d} f(x)\mathrm{d}\mu(x) = \int_{\mathbb{S}^d} f(Ax)\mathrm{d}\mu(x).$$

Let $X = (X_1, \dots, X_{d+1})$ be a vector of independent centered and Gaussian random variables.

a) Prove that the random variable $U = X/\|X\|_{L^2}$ is uniformly distributed on the sphere.

b) Prove that, as $d \to \infty$, the main part of the globe is concentrated close to the Equator, i.e. for any $\varepsilon > 0$,

$$\int_{x\in\mathbb{S}^d, |x_1|<\epsilon} \mathrm{d}\mu(x) \to 1.$$

Exercise 5. Let (X_1, X_2) be a Gaussian vector with mean (m_1, m_2) and non-degenerate covariance matrix $(C_{ij})_{1\leq i,j\leq 2}$. Prove that

$$\mathbb{E}[X_1 \mid X_2] = m_1 + \frac{C_{12}}{C_{22}}(X_2 - m_2).$$

Exercise 6. Let X be a random variable such that $\mathbb{P}(X > t) = \exp(-t)$ for any $t \geq 0$. Let $Y = \min(X, s)$, where $s > 0$ is fixed. Prove that, almost surely,

$$\mathbb{E}[X \mid Y] = Y\mathbb{1}_{Y<s} + (1+s)\mathbb{1}_{Y=s}.$$

Exercise 7. Let μ and ν be two probability measures such that $\mu \ll \nu$ and $\nu \ll \mu$ (usually abbreviated $\mu \sim \nu$). Let $X = \frac{\mathrm{d}\mu}{\mathrm{d}\nu}$.

(i) Prove that $\nu(X = 0) = 0$.

(ii) Prove that $\frac{1}{X} = \frac{\mathrm{d}\nu}{\mathrm{d}\mu}$ almost surely (for μ or ν).

Exercise 8. Let $(X_n)_{n\geq 0}$ be defined on $(\Omega, \mathcal{A}, \mathbb{P})$. Assume this sequence converges in probability (under $\mathbb{P}$) to X. Let $\mathbb{Q}$ be another probability measure on $(\Omega, \mathcal{A})$ assumed to be absolutely continuous w.r.t. $\mathbb{P}$. Prove that $X_n \to X$ in probability under $\mathbb{Q}$.