Exercise 1. Let $(X_i)_{i\geq 1}$ be i.i.d. Gaussian with mean 1 and variance 3. Show that

$$\lim_{n\to\infty} \frac{X_1+\cdots+X_n}{X_1^2+\cdots+X_n^2} = \frac{1}{4} \text{ a.s.}$$

Exercise 2. Let f be a continuous function on $[0,1]$. Calculate the asymptotics, as $n\to\infty$, of

$$\int_{[0,1]^n} f\left(\frac{x_1+\cdots+x_n}{n}\right) \mathrm{d}x_1 \ldots \mathrm{d}x_n.$$

Exercise 3. Assume $(\Omega, \mathcal{A}, \mathbb{P})$ is such that Ω is countable and $\mathcal{A} = 2^\Omega$. Prove that convergence in probability and convergence almost sure are the same.

Exercise 4. Let $\alpha > 0$ and, given $(\Omega, \mathcal{A}, \mathbb{P})$, let $(X_n, n\geq 1)$ be a sequence of independent real random variables with law $\mathbb{P}(X_n = 1) = \frac{1}{n^\alpha}$ and $\mathbb{P}(X_n = 0) = 1 - \frac{1}{n^\alpha}$. Prove that $X_n \to 0$ in $\mathcal{L}^1$, but that almost surely

$$\limsup_{n\to\infty} X_n = \begin{cases} 1 & \text{if} \quad \alpha \leq 1 \\ 0 & \text{if} \quad \alpha > 1 \end{cases}.$$

Exercise 5. Let $(X_n)_{n\geq 0}$ be real, independent, random variables on $(\Omega, \mathcal{A}, \mathbb{P})$.

a) Prove that the radius of convergence R of the random series $\sum_{n\geq 0} X_n z^n$ is almost surely constant.

b) Assume also that the X_n's have the same distribution. Prove that $R = 0$ a.s. if $\mathbb{E}[\log(|X_0|)_+] = \infty$, and $R \geq 1$ a.s. if $\mathbb{E}[\log(|X_0|)_+] < \infty$.

Exercise 6. Prove that there is no probability measure on $\mathbb{N}$ such that for any $n \geq 1$, the probability of the set of multiples of n is $1/n$.

Long problem. The goal of this problem is to prove the iterated logarithm law, first for Gaussian random variables. In other words, for $X_1, X_2 \ldots$ i.i.d. standard Gaussian random variables, denoting $S_n = X_1 + \cdots + X_n$, we have

$$\mathbb{P}\left(\limsup_{n\to\infty} \frac{S_n}{\sqrt{2n\log\log n}} = 1\right) = 1 \tag{0.1}$$

(1) Prove that

$$\mathbb{P}(X_1 > \lambda) \underset{\lambda\to\infty}{\sim} \frac{1}{\lambda\sqrt{2\pi}} e^{-\frac{\lambda^2}{2}}.$$

In the following questions we denote $f(n) = \sqrt{2n\log\log n}$, $\lambda > 1$, $c, \alpha > 0$, $A_k = \{S_{\lfloor \lambda^k \rfloor} \geq cf(\lambda^k)\}$, $C_k = \{S_{\lfloor \lambda^{k+1} \rfloor} - S_{\lfloor \lambda^k \rfloor} \geq cf(\lambda^{k+1} - \lambda^k)\}$ and $D_k = \{\sup_{n\in [\![\lambda^k, \lambda^{k+1}]\!]} \frac{S_n - S_{\lfloor \lambda^k \rfloor}}{f(\lambda^k)} \geq \alpha\}$.

(2) Prove that for any $c > 1$ we have $\sum_{k\geq 1} \mathbb{P}(A_k) < \infty$ and

$$\limsup_{k\to\infty} \frac{S_{\lfloor \lambda^k \rfloor}}{f(\lambda^k)} \leq 1 \text{ a.s.}$$

(3) Prove that for any $c<1$ we have $\sum_{k\geq 1}\mathbb{P}(C_k)=\infty$ and

$$\mathbb{P}(C_k \text{ i.o.})=1.$$

(4) Let $\varepsilon>0$ and choose $c=1-\varepsilon/10$. Prove that almost surely the following inequality holds for infinitely many k:

$$\frac{S_{\lfloor\lambda^{k+1}\rfloor}}{f(\lambda^{k+1})}\geq c\frac{f(\lambda^{k+1}-\lambda^k)}{f(\lambda^{k+1})}-(1+\varepsilon)\frac{f(\lambda^k)}{f(\lambda^{k+1})}.$$

(5) By choosing a large enough λ in the previous inequality, prove that almost surely

$$\limsup_{n\to\infty}\frac{S_n}{f(n)}\geq 1.$$

(6) Prove that for any $n\in[\![\lambda^k,\lambda^{k+1}]\!]$ and $S_n>0$ we have

$$\frac{S_n}{f(n)}\leq\frac{S_{\lfloor\lambda^k\rfloor}}{f(\lfloor\lambda^k\rfloor)}+\frac{S_n-S_{\lfloor\lambda^k\rfloor}}{f(\lfloor\lambda^k\rfloor)}.$$

(7) Prove that

$$\mathbb{P}(D_k)\underset{k\to\infty}{\sim}2\mathbb{P}\left(X_1\geq\frac{\alpha f(\lambda^k)}{\sqrt{\lambda^{k+1}-\lambda^k}}\right)\underset{k\to\infty}{\sim}\frac{\mathrm{cst}}{\sqrt{\log k}}\left(\frac{1}{k}\right)^{\frac{\alpha^2}{\lambda-1}}.$$

(8) Prove that for $\alpha^2>\lambda-1$, almost surely

$$\limsup_{n\to\infty}\frac{S_n}{f(n)}\leq\limsup_{n\to\infty}\frac{S_{\lfloor\lambda^k\rfloor}}{f(\lambda^k)}+\alpha.$$

(9) By choosing appropriate λ and α, prove that almost surely

$$\limsup_{n\to\infty}\frac{S_n}{f(n)}\leq 1.$$

(10) State a result similar to (0.1) for i.i.d. uniformly bounded random variables. Which steps in the above proof need to be modified to prove this universality result? How?