Roland Bauerschmidt *Fall 2024*

Exercise 1. Prove that if a sequence of real random variables (X_n) converge in distribution to X, and (Y_n) converges in distribution to a constant c, then $X_n + Y_n$ converges in distribution to $X + c$.

Exercise 2. Assume that (X, Y) has joint density

$$ce^{-(1+x^2)(1+y^2)},$$

where c is properly chosen. Are X and Y Gaussian random variables? Is (X, Y) a Gaussian vector? [A Gaussian vector on $\mathbb{R}^n$ has probability density function proportional to $e^{-\frac{1}{2}x^T C^{-1} x}$ for some positive definite matrix C (or a is a weak limit of such distributions).]

Exercise 3. The goal of this exercise is to prove that any function, continuous on an interval of $\mathbb{R}$, can be approximated by polynomials, arbitrarily close for the L^∞ norm (this is Bernstein's proof of the Weierstrass approximation theorem). Let f be a continuous function on $[0, 1]$. The n-th Bernstein polynomial is

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right).$$

a) Let $S_n(x) = B^{(n,x)}/n$, where $B^{(n,x)}$ is a binomial random variable with parameters n and x: $B^{(n,x)} = \sum_{\ell=1}^{n} X_i$ where the X_i's are independent and $\mathbb{P}(X_i = 1) = x$, $\mathbb{P}(X_i = 0) = 1 - x$. Prove that $B_n(x) = \mathbb{E}(f(S_n(x)))$.

b) Prove that $\|B_n - f\|_{L^\infty([0,1])} \to 0$ as $n \to \infty$.

Exercise 4. You toss a coin repeatedly and independently. The probability to get a head is p, a tail is $1 - p$. Let A_k be the following event: k or more consecutive heads occur amongst the tosses numbered $2^k, \dots, 2^{k+1} - 1$. Prove that $\mathbb{P}(A_k \text{ i.o.}) = 1$ if $p \geq 1/2$, 0 otherwise.

Here, i.o. stands for "infinitely often", and A_k i.o. is the event $\cap_{n\geq 1} \cup_{m\geq n} A_m$.

Exercise 5. Prove the Central Limit Theorem using Lindeberg's exchange method: Let X_i be i.i.d. random variables with mean 0 and variance 1 and Z_i be i.i.d. standard normal random variables. For a bounded continuous function f, then consider

$$E[f(\frac{X_1 + \cdots + X_n}{\sqrt{n}})] - E[f(\frac{Z_1 + \cdots + Z_n}{\sqrt{n}})]. \tag{0.1}$$

Then replace one X_i by Z_i on the left-hand side at a time and estimate the error using Taylor's formula.

Exercise 6. For any probability measure μ supported on $[0, \infty)$, one defines the Laplace transform as

$$\mathscr{L}_\mu(\lambda) = \int_0^\infty e^{-\lambda x} \mathrm{d}\mu(x), \ \lambda \geq 0.$$

(1) Prove that $\mathscr{L}_\mu$ is well-defined, continuous on $[0, \infty)$ and C^∞ on $(0, \infty)$.
(2) Prove that $\mathscr{L}_\mu$ characterizes the probability measure μ supported on $[0, \infty)$.

(3) Assume that for a sequence $(\mu_n)_{n\geq 1}$ of probability measure supported on $[0,\infty)$, one has $\mathscr{L}_{\mu_n}(\lambda) \to \ell(\lambda)$ for any $\lambda \geq 0$, and ℓ is right-continuous at 0. Prove that $(\mu_n)_{n\geq 1}$ is tight, and that it converges weakly to a measure μ such that $\ell = \mathscr{L}_\mu$.