

**Exercise 1.** Let  $(X_n)_{n \geq 1}$  be independent Gaussian such that  $\mathbb{E}(X_i) = m_i$ ,  $\text{var}(X_i) = \sigma_i^2$ ,  $i \geq 1$ . Let  $S_n = \sum_{i=1}^n X_i$  and  $\mathcal{F}_n = \sigma(X_i, 1 \leq i \leq n)$ .

a) Find sequences  $(b_n)_{n \geq 1}$ ,  $(c_n)_{n \geq 1}$  of real numbers such that  $(S_n^2 + b_n S_n + c_n)_{n \geq 1}$  is a  $(\mathcal{F}_n)_{n \geq 1}$ -martingale.

b) Assume moreover that there is a real number  $\lambda$  such that  $e^{\lambda X_i} \in L^1$  for any  $i \geq 1$ . Find a sequence  $(a_n^{(\lambda)})_{n \geq 1}$  such that  $(e^{\lambda S_n - a_n^{(\lambda)}})_{n \geq 1}$  is a  $(\mathcal{F}_n)_{n \geq 1}$ -martingale.

**Exercise 2.** Let  $(X_k)_{k \geq 0}$  be i.i.d. random variables,  $\mathcal{F}_m = \sigma(X_1, \dots, X_m)$  and  $Y_m = \prod_{k=1}^m X_k$ . Under which conditions is  $(Y_m)_{m \geq 1}$  a  $(\mathcal{F}_m)_{m \geq 1}$ -submartingale, supermartingale, martingale?

**Exercise 3.** Let  $(\mathcal{F}_n)_{n \geq 0}$  be a filtration,  $(X_n)_{n \geq 0}$  a sequence of integrable random variables with  $\mathbb{E}(X_n | \mathcal{F}_{n-1}) = 0$ , and assume  $X_n$  is  $\mathcal{F}_n$ -measurable for every  $n$ . Let  $S_n = \sum_{k=0}^n X_k$ . Show that  $(S_n)_{n \geq 0}$  is a  $(\mathcal{F}_n)_{n \geq 0}$ -martingale.

**Exercise 4.** Let  $a > 0$  be fixed,  $(X_i)_{i \geq 1}$  be iid,  $\mathbb{R}^d$ -valued random variables, uniformly distributed on the ball  $B(0, a)$ . Set  $S_n = x + \sum_{i=1}^n X_i$ .

- (i) Let  $f$  be a superharmonic function. Show that  $(f(S_n))_{n \geq 1}$  defines a supermartingale.
- (ii) Prove that if  $d \leq 2$  any nonnegative superharmonic function is constant. Does this result remain true when  $d \geq 3$ ?

**Exercise 5.** Let  $(S_n)_{n \geq 0}$  be a  $(\mathcal{F}_n)$ -martingale and  $\tau$  a stopping time with finite expectation. Assume that there is a  $c > 0$  such that, for all  $n$ ,  $\mathbb{E}(|S_{n+1} - S_n| | \mathcal{F}_n) < c$ .

Prove that  $(S_{\tau \wedge n})_{n \geq 0}$  is a uniformly integrable martingale, and that  $\mathbb{E}(S_\tau) = \mathbb{E}(S_0)$ .

Consider now the random walk  $S_n = \sum_{k=1}^n X_k$ , the  $X_k$ 's being iid,  $\mathbb{P}(X_1 = 1) = \mathbb{P}(X_1 = -1) = 1/2$ . For some  $a \in \mathbb{N}^*$ , let  $\tau = \inf\{n | S_n = -a\}$ . Prove that

$$\mathbb{E}(\tau) = \infty.$$

**Exercise 6.** Let  $X$  be a standard random walk in dimension 1, and for any positive integer  $a$ ,  $\tau_a = \inf\{n \geq 0 | X_{\tau_a} = a\}$ . For any  $\theta > 0$ , calculate

$$\mathbb{E}((\cosh \theta)^{-\tau_a}).$$

**Exercise 7.** Let  $N_n$  be the size of a population of bacteria at time  $n$ . At each time each bacterium produces a number of offspring and dies. The number of offspring is independent for each bacterium and is distributed according to the Poisson law with rate parameter  $\lambda = 2$ . Assuming that  $N_1 = a > 0$ , find the probability that the population will eventually die, i.e., find  $\mathbb{P}(\{\text{there is } n \text{ such that } N_n = 0\})$ .

**Exercise 8.** Let  $X_n, n \geq 0$ , be iid complex random variables such that  $\mathbb{E}(X_1) = 0, 0 < \mathbb{E}(|X_1|^2) < \infty$ . For some parameter  $\alpha > 0$ , let

$$S_n = \sum_{k=1}^n \frac{X_k}{k^\alpha}.$$

Prove that if  $\alpha > 1/2$ ,  $S_n$  converges almost surely. What if  $0 < \alpha \leq 1/2$  ?

**Exercise 9.** Let  $(Y_n)_{n \in \mathbb{N}^*}$  be a sequence of random variables, and assume  $(Y_n)$  converges in distribution to a limiting  $Y$ . Also, on some probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ , the sequence of independent random variables  $X := (X_n)_{n \in \mathbb{N}^*}$  is defined, and we assume that the sequence of partial sums  $(S_n)_{n \in \mathbb{N}}$  (i.e.  $S_0 = 0$  and  $S_n := \sum_{j=1}^n X_j$ ) converges in distribution. Set  $(\mathcal{F}_n)$  the natural filtration of  $X$  and  $\Phi_n(t) = \mathbb{E}(\exp(itS_n))$  for  $t \in \mathbb{R}$ .

- (i) Establish that  $(\Phi_{Y_n}(\cdot))_{n \geq 1}$  converges uniformly on every compact, i.e. show that for any  $a > 0$ ,  $\max_{t \in [-a, a]} |\Phi_{Y_n}(t) - \Phi_Y(t)| \rightarrow 0$  as  $n \rightarrow \infty$ . Establish moreover that there exists  $a > 0$  such that for any  $n \geq 1$ ,  $\min_{t \in [-a, a]} |\Phi_{Y_n}(t)| \geq 1/2$ .
- (ii) Show that there exists  $t_0 > 0$  such that if  $t \in [-t_0, t_0]$ , then  $(\exp(itS_n)/\Phi_n(t))_{n \geq 0}$  is a  $(\mathcal{F}_n)$ -martingale (i.e. both its real and imaginary parts are martingales).
- (iii) Prove that we can choose  $t_0 > 0$  such that for any  $t \in [-t_0, t_0]$ ,  $\lim_{n \rightarrow \infty} \exp(itS_n)$  exists  $\mathbb{P}$ -a.s.
- (iv) Set

$$C = \{(t, \omega) \in [-t_0, t_0] \times \Omega : \lim_{n \rightarrow \infty} \exp(itS_n(\omega)) \text{ exists}\}.$$

Prove that  $C$  is measurable, i.e. in the product of  $\mathcal{B}([-t_0, t_0])$  with  $\mathcal{F}$ .

- (v) Establish that  $\int_{-t_0}^{t_0} \mathbf{1}_C(t, \omega) \mathbb{P}(d\omega) dt = 2t_0$ .
- (vi) Prove that  $\lim_{n \rightarrow \infty} S_n$  exists  $\mathbb{P}$ -a.s.