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Exercise 1. Let $(\mathcal{G}_\alpha)_{\alpha\in A}$ be an arbitrary family of σ-fields defined on an abstract space Ω, with A possibly uncountable. Show that $\cap_{\alpha\in A}\mathcal{G}_\alpha$ is also a σ-field.

Exercise 2. Let $\varnothing \subsetneq A \subsetneq B \subsetneq \Omega$ (these are strict inclusions). What is the σ-field generated by $\{A, B\}$?

Exercise 3. Let $\mathcal{F}$, $\mathcal{G}$ be σ-fields for the same Ω. Is $\mathcal{F}\cup\mathcal{G}$ a σ-field?

Exercise 4. For $\Omega = \mathbb{N}$ and $n \geq 0$, let $\mathcal{F}_n = \sigma(\{\{0\},\dots,\{n\}\})$. Show that $(\mathcal{F}_n)_{n\geq 0}$ is a non-decreasing sequence but that $\cup_{n\geq 0}\mathcal{F}_n$ is a not a σ-field.

Exercise 5. Let Ω be an infinite set (countable or not). Let $\mathcal{A}$ be the set of subsets of Ω that are either finite or with finite complement in Ω. Prove that $\mathcal{A}$ is a field but not a σ-field.

Exercise 6. Can you build an infinite, countable σ-field?

Exercise 7. Prove Dynkin's π-system lemma.

Exercise 8. Let $\mathbb{P}$ be a probability measure on Ω, endowed with a σ-field $\mathcal{A}$.

(i) What is the meaning of the following events, where all $A_n \in \mathcal{A}$?

$$\liminf_{n\to\infty} A_n = \bigcup_{n\geq 1}\bigcap_{k\geq n} A_k, \quad \limsup_{n\to\infty} A_n = \bigcap_{n\geq 1}\bigcup_{k\geq n} A_k.$$

(ii) Prove that $\limsup_{n\to\infty} A_n$ and $\liminf_{n\to\infty} A_n$ are in $\mathcal{A}$.

(iii) In the special case $\Omega = \mathbb{R}$, for any $p \geq 1$, let

$$A_{2p} = \left[-1, 2 + \frac{1}{2p}\right), \quad A_{2p+1} = \left(-2 - \frac{1}{2p+1}, 1\right].$$

What are $\liminf_{n\to\infty} A_n$ and $\limsup_{n\to\infty} A_n$?

(iv) Prove that the following always holds:

$$\mathbb{P}\left(\liminf_{n\to\infty} A_n\right) \leq \liminf_{n\to\infty} \mathbb{P}\left(A_n\right), \mathbb{P}\left(\limsup_{n\to\infty} A_n\right) \geq \limsup_{n\to\infty} \mathbb{P}\left(A_n\right).$$

Exercise 9. The symmetric difference of two events A and B, denoted $A\triangle B$, is the event that precisely one of them occurs: $A\triangle B = (A\cup B)\backslash(A\cap B)$.

(i) Write a formula for $A\triangle B$ that only involves the operations of union, intersection and complement, but no set difference.

(ii) Define $d(A,B) = \mathbb{P}(A\triangle B)$. Show that for any three events A, B, C,

$$d(A,B) + d(B,C) - d(A,C) = 2\left(\mathbb{P}\left(A\cap B^c\cap C\right) + \mathbb{P}\left(A^c\cap B\cap C^c\right)\right).$$

(iii) Assume $A\subset B\subset C$. Prove that $d(A,C) = d(A,B) + d(B,C)$.

Exercise 10. Prove the Bonferroni (inclusion–exclusion) inequalities: if $A_i \in \mathcal{A}$ is a sequence of events, then

(i) $\mathbb{P}\left(\cup_{i=1}^n A_i\right) \leq \sum_{i=1}^n \mathbb{P}(A_i)$,
(ii) $\mathbb{P}\left(\cup_{i=1}^n A_i\right) \geq \sum_{i=1}^n \mathbb{P}(A_i) - \sum_{i<j} \mathbb{P}(A_i \cap A_j)$,
(iii) $\mathbb{P}\left(\cup_{i=1}^n A_i\right) \leq \sum_{i=1}^n \mathbb{P}(A_i) - \sum_{i<j} \mathbb{P}(A_i \cap A_j) + \sum_{i<j<k} \mathbb{P}(A_i \cap A_j \cap A_k)$.